Announcements and Such

- Today's Music: *Michael Hedges*
- HW #4 is due today @ 4pm, usual drill (chapter 4 — proofs).
- I've posted HW #5, which is due next Thursday @ 4pm.
  - A few LMPL symbolization problems (chapter 5).
  - Mostly, working with LMPL Interpretations (chapter 6).

☞ I've posted a new handout entitled "Working with LMPL Interpretations", which I will be going over in class very soon.

- Today: Chapter 5 (rather quickly), and then Chapter 6 (Intro.)
  - Finishing-up our discussion of symbolizations into LMPL.
  - Introduction to LMPL Semantics (working with LMPL interpretations).

New Elements of LMPL

- Now, upper-case letters can be used as LSL sentences *or* **predicates**.
  - *E.g.*, the predicate 'is tall' can be symbolized using 'T'.
- Lower-case letters 'a'–'s' will be used as **individual constants** (**names**).
- This yields **new atomic sentences** with *subject-predicate structure*:
  - *E.g.*, 'Branden is tall' $\mapsto$ 'Tb'.
- We also have two **quantifier phrases**: **all** ($\forall$) and **some** ($\exists$).
- Lower-case letters 't'–'z' will be used as **variables**.
- We use **variables** + **quantifiers** to symbolize **general claims**.
  - *E.g.*, 'Someone is wise'
    $\mapsto$ 'There exists an x such that x is wise.'
    $\mapsto$ '$(\exists x)Wx$'
- Each general claim quantifies over a **domain/universe of discourse**.

Symbolization in LMPL VI: More Examples with ∃

- Let's symbolize the following sentences. Whenever we symbolize in LMPL, we must state our dictionary of monadic predicates, and we must also say what the domain of discourse is over which we are quantifying.
  1. No smoggy city is unpolluted.
  2. Vampires do not exist.
  3. If ghosts and vampires do not exist, then nothing can be a ghost without being a vampire.
- If the dictionary is (where the domain is people in this classroom now):

  $S\_\_$: __ is standing up at the podium.
  $W\_\_$: __ is wealthy.
  b : Branden

  then what do the following two LMPL sentences assert (in English)?

  $\sim(\exists x)(Sx \,\&\, Wx)$ $\qquad\qquad \sim Wb$

Symbolization in LMPL VII: Back to ① and ②

- Now, we are in a position to symbolize in LMPL the argument ① that we saw at the beginning of this lecture:

$$①_{\text{LMPL}} \quad \begin{array}{l} Ws \\ \therefore (\exists x)Wx \end{array}$$

- Since there are only finitely many people, we can see why this argument is valid, by representing its conclusion as a long (but finite!) disjunction, in which its only premise is a disjunct:

$$① \quad \begin{array}{l} Ws \\ \therefore Wa \vee \cdots \vee Ws \vee \cdots \end{array}$$

- We can use a similar trick for argument ②. In that case, it's premise [‘$(\forall x)Hx$’] entails a conjunction [‘$Ha \,\&\, \cdots \,\&\, Hp \,\&\, \cdots$’], and its conclusion [‘Hp’] is one of the conjuncts of that conjunction.

Some Symbolizations Involving ∃

E__ : __ is an even number a : the number 2
P__ : __ is a prime number Domain : natural numbers ($\mathbb{N}$)
G__ : __ is greater than the number 2

(1) There exists a prime number and there exists an even number.

$(\exists x)Px \,\&\, (\exists x)Ex$

(2) There exists an even prime number. [$(\exists x)(Px \,\&\, Ex)$]

(3) 2 is an even prime number. [$Ea \,\&\, Pa$]

(4) If 2 is prime, then there are some even primes. [$Pa \to (\exists x)(Px \,\&\, Ex)$]

(5) No number is even if it is prime. [$\sim(\exists x)(Px \,\&\, Ex)$]

- Careful with this one! Why *isn't* this '$\sim(\exists x)(Px \to Ex)$'?
- Compare: No number is even if it is prime and greater than 2.
  - In LMPL, this is: '$\sim(\exists x)[(Px \,\&\, Gx) \,\&\, Ex]$', which is *true*. *Why?*
  - Note: '$\sim(\exists x)[(Px \,\&\, Gx) \to Ex]$' is *false*! *Why?*

The *Universal* Quantifier $\forall$

- To symbolize English sentences like ‘Everyone is happy’, we will need the *universal* quantifier ‘$\forall$’ (which means ‘every’ or ‘all’).
  (*i*) We begin with the raw English sentence: ‘Everyone is happy’.
  (*ii*) Then, we move to the *Logish* form: ‘For every x, x is happy’.
  (*iii*) Finally, we have the full LMPL symbolization: ‘$(\forall x)Hx$’.
- As with the existential quantifier, we must be careful with the *scope* of ‘$\forall$’. How would we symbolize the following two sentences?
  (1) ‘Everyone is happy and everyone is wise.’
  (2) ‘Everyone is happy and wise.’
- These sentences get symbolized differently, because they have different (syntactic) *structures*. But, do they have different *meanings*? In Chapter 6, we’ll *prove* the answer to this question.

The Universal Quantifier II

- How should one symbolize the following English sentence?

  (3) 'Everyone who is happy is wise.'

- Note: Unlike (1) and (2) above, (3) does *not* have the consequence that *everyone* is happy. So, what, exactly, *does* (3) say?

- (3) says that *if* a person is happy, *then* that person is wise. This suggests the following *Logish* form (*wrt* the domain of people):

  'For every x, if x is happy then x is wise.'

- Now, we are ready for the full LMPL symbolization:

$$(\forall x)(Hx \rightarrow Wx)$$

- We will use this same trick to symbolize sentences like 'Every happy person is a wise person' or 'If someone is happy then he/she is wise', which both assert the same thing as (3).

The Universal Quantifier III

- How should one symbolize the following English sentence?

  (4) ‘Only happy people are wise.’

- Note: (4) does *not* say that *all* happy people are wise. That is, unlike (3), (4) does *not* say that a person is wise *if* he/she is happy. Rather, (4) says that a person is wise *only if* he/she is happy.
- This suggests the following *Logish* form (domain of people):

  ‘For every x, x is wise *only if* x is happy.’

- Now, we are ready for the full LMPL symbolization:

$$(\forall x)(Wx \to Hx)$$

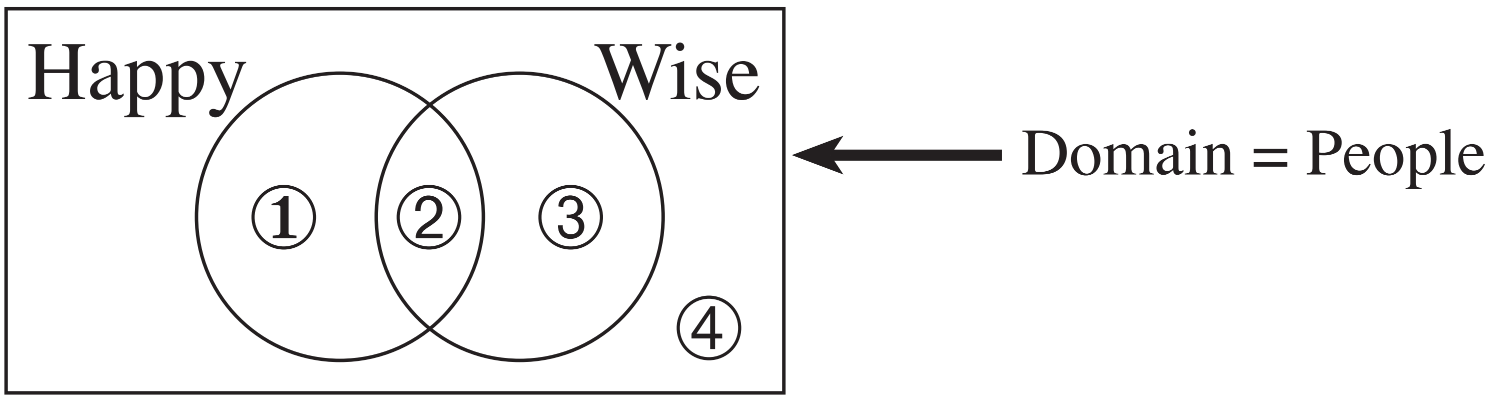
- Here, we have the usual distinction between necessary and sufficient conditions. (3) says that happiness is *sufficient* for wisdom. But, (4) says that happiness is *necessary* for wisdom.

The Universal Quantifier IV, and Venn Diagrams

- Consider the following English sentence:

  (5) ‘No one who is unhappy is wise.’

- When trying to paraphrase or symbolize sentences like this in LMPL, it is useful to *picture* what they say using a *Venn Diagram*:



- (5) says that region ③ in the Venn Diagram is empty. So, (5) asserts the same thing as the following LMPL sentence:

  (5.1) $(\forall x)(Wx \to Hx)$

The Intimate Relationship Between $\exists$ and $\forall$

- What we have just shown (informally) is:

  '$\sim(\exists x)(Wx \,\&\, {\sim}Hx)$' *is equivalent to* '$(\forall x)(Wx \rightarrow Hx)$'

- This is just a *special case* of the following *general equivalences*:

  $\ulcorner{\sim}(\exists \nu){\sim}\phi\nu\urcorner$ *is equivalent to* $\ulcorner(\forall \nu)\phi\nu\urcorner$

  and

  $\ulcorner{\sim}(\forall \nu){\sim}\phi\nu\urcorner$ *is equivalent to* $\ulcorner(\exists \nu)\phi\nu\urcorner$

- Here, 'ϕ' is a *metavariable* ranging over formulas of LMPL (thought of as functions of ν), and 'ν' ranges over variable symbols of LMPL.
- It follows from the second general equivalence above that $\sim(\exists x)(Wx \,\&\, {\sim}Hx)$ is equivalent to $\sim\sim(\forall x){\sim}(Wx \,\&\, {\sim}Hx)$. But, this is equivalent to $(\forall x){\sim}(Wx \,\&\, {\sim}Hx)$, hence $(\forall x)(Wx \rightarrow Hx)$.
- Our formal semantics will make these relationships more precise.

- Here's *why* (informally) $\ulcorner{\sim}(\exists v){\sim}\phi v\urcorner$ and $\ulcorner(\forall v)\phi v\urcorner$ are equivalent.
- Start with the existential claim inside the negation $\ulcorner{\sim}(\exists v){\sim}\phi v\urcorner$:

$$(\exists v){\sim}\phi v$$

- Next, note that, informally, $(\exists v){\sim}\phi v$ asserts a *disjunction*:

$${\sim}\phi a \vee {\sim}\phi b \vee \ldots$$

- So, by DeMorgan, its negation '${\sim}(\exists v){\sim}\phi v$' asserts a *conjunction*:

$${\sim}{\sim}\phi a \,\&\, {\sim}{\sim}\phi b \,\&\ldots$$

- Then, by Double Negation (DN), we can see this is equivalent to:

$$\phi a \,\&\, \phi b \,\&\ldots$$

- But, this just asserts that *every* individual has ϕ. In other words, this says the same thing that the universal claim $(\forall v)\phi v$ says!
- Therefore, $\ulcorner{\sim}(\exists v){\sim}\phi v\urcorner$ *is equivalent to* $\ulcorner(\forall v)\phi v\urcorner$. *QED*.
- We can run a parallel argument for $\ulcorner{\sim}(\forall v){\sim}\phi v\urcorner$ and $\ulcorner(\exists v)\phi v\urcorner$.

Further Symbolization Problems

- If someone says "all athletes are not superstars" (another example: "all that glitters is not gold"), they are not to be symbolized exactly as read.
  - Sounds like $(\forall x)(Ax \rightarrow \sim Sx)$, but it's really $\sim(\forall x)(Ax \rightarrow Sx)$.
  - Note: this is equivalent to $(\exists x)(Ax \,\&\, \sim Sx)$.
- "The only" gets symbolized like "All". Example:
  - "The only animals in this canyon are skunks" is $(\forall x)((Ax \,\&\, Cx) \rightarrow Sx)$. Where Ax: x is an animal, Cx: x is in this canyon, and Sx: x is a skunk.
  - Clearly, $(\forall x)(Sx \rightarrow (Ax \,\&\, Cx))$ is *not* what's intended. Why?
- "None but", "none except" and "no ... except" are like "Only". Examples:
  - "None but the brave deserve a Purple Heart" is $(\forall x)(Px \rightarrow Bx)$. Where Bx: x is brave, Px: x deserves a Purple Heart.
  - "No birds except peacocks are proud of their tails" is equivalent to "Only peacocks are birds that are proud of their tails".

Further Symbolizations Involving $\forall$ and $\exists$

- How should we paraphrase and/or symbolize the following sentence?

  (6) If anyone is wealthy, then economists are.

- At first blush, we might try to paraphrase (6) as follows:

  (6.1) If everyone is wealthy, then all economists are wealthy (which gives the LMPL symbolization: '$(\forall x)Wx \rightarrow (\forall x)(Ex \rightarrow Wx)$').

- But, (6.1) *cannot* be right. If the antecedent of (6.1) is true, then *everybody* is wealthy (not just the economists!). In this sense, (6.1) is analogous to an LSL *tautology* — it's true *in all possible worlds*. Is *that* all (6) asserts?

- In fact, (6) asserts something *much stronger* than (6.1). What (6) says is that all it takes for every economist to be wealthy is for there to exist *one* wealthy person. This leads to the following alternative paraphrase of (6):

  (6.2) If *someone* is wealthy, then all economists are wealthy (which gives the LMPL symbolization: '$(\exists x)Wx \rightarrow (\forall x)(Ex \rightarrow Wx)$').

Still More Symbolizations Involving ∀ and ∃

- How should we paraphrase and/or symbolize the following sentence?

  (7) Every wealthy logician is happy.

- It helps to do a *Logish*, intermediate form first:

  (7.1) For every x, if x is wealthy and x is a logician, then x is happy.

- This leads to the following LMPL symbolization:

  (7.2) $(\forall x)((Wx \& Lx) \rightarrow Hx)$

- OK, but what about the following sentence?

  (8) No wealthy economists are happy.

- This time, the *Logish*, intermediate form is:

  (8.1) Not: there is at least one x such that x is wealthy, and x is an economist, and x is happy.

- Which leads to the following LMPL symbolization:

  (8.2) $\sim(\exists x)((Wx \& Ex) \& Hx)$

One Last Symbolization Involving ∀

(9) A fetus is a person, but an embryo is not.

- In this case, the domain of discourse must be *wider* than the domain of people (since we need to be able to say that some things are *not* persons). And, 'is a person' must then be included as a *predicate* in our dictionary.

P__ : __ is a person	F__ : __ is a fetus
E__ : __ is an embryo	Domain of Discourse : *all things*

- Now, it helps to do a *Logish*, intermediate form first:

  (9.1) For every x, if x is a fetus then x is a person, and for every x, if x is an embryo then x not a person.

- This leads to the following LMPL symbolization:

  (9.2) $(\forall x)(Fx \rightarrow Px) \,\&\, (\forall x)(Ex \rightarrow \sim Px)$

  which is *semantically equivalent* (as we will *prove* in Chapter 6) to:

  (9.3) $(\forall x)((Fx \rightarrow Px) \,\&\, (Ex \rightarrow \sim Px))$

  But, (9.2) is *preferred* over (9.3), since (9.2) is closer to the *structure* of (9).

LMPL Symbolizations: Summary and Tips

- Some general symbolization forms we've seen so far:
  - All Fs are Gs. LMPL: $(\forall x)(Fx \to Gx)$.
  - An F is a G. LMPL: $(\forall x)(Fx \to Gx)$.
  - Fs are Gs. LMPL: $(\forall x)(Fx \to Gx)$.
  - Only Fs are Gs. LMPL: $(\forall x)(Gx \to Fx)$.
  - The only Fs are Gs. LMPL: $(\forall x)(Fx \to Gx)$.
  - Some Fs are Gs. LMPL: $(\exists x)(Fx \,\&\, Gx)$.
  - No Fs are Gs. LMPL: $\sim(\exists x)(Fx \,\&\, Gx)$.
  - Nothing is an F if it's G. $\sim(\exists x)(Gx \,\&\, Fx)$. [**NOT** $\sim(\exists x)(Gx \to Fx)$!]
  - If anything is an F, then Gs are. LMPL: $(\exists x)Fx \to (\forall x)(Gx \to Fx)$.
  - 'All Fs are not Gs' can sometimes *really* be $\sim(\forall x)(Fx \to Gx)$.
  - None but Fs are Gs (or None except Fs are Gs). $(\forall x)(Gx \to Fx)$.
- Remember: $\ulcorner\sim(\exists v)\sim\phi v\urcorner$ *is equivalent to* $\ulcorner(\forall v)\phi v\urcorner$ and $\ulcorner\sim(\forall v)\sim\phi v\urcorner$ *is equivalent to* $\ulcorner(\exists v)\phi v\urcorner$. You should be able to use these proficiently.

- Some equivalences:
  - 'All Fs are Gs' is equivalent to 'No Fs are non-Gs'.
    - $(\forall x)(Fx \to Gx)$ is equivalent to $\sim(\exists x)(Fx \,\&\, {\sim}Gx)$.
  - 'All Fs are Gs' is equivalent to 'All non-Gs are non-Fs'.
    - $(\forall x)(Fx \to Gx)$ is equivalent to $(\forall x)({\sim}Gx \to {\sim}Fx)$.
  - 'Some Fs are Gs' is equivalent to 'Some Gs are Fs'.
    - $(\exists x)(Fx \,\&\, Gx)$ is equivalent to $(\exists x)(Gx \,\&\, Fx)$.
  - 'No Fs are Gs' is equivalent to 'No Gs are Fs'.
    - $\sim(\exists x)(Fx \,\&\, Gx)$ is equivalent to $\sim(\exists x)(Gx \,\&\, Fx)$.
- Some *non*-equivalences:
  - 'All Fs are Gs' is *not* equivalent to 'All Gs are Fs'.
    - $(\forall x)(Fx \to Gx)$ is *not* equivalent to $(\forall x)(Gx \to Fx)$.
  - 'Some Fs are non-Gs' is *not* equivalent to 'Some Gs are non-Fs'.
    - $(\exists x)(Fx \,\&\, {\sim}Gx)$ is *not* equivalent to $(\exists x)(Gx \,\&\, {\sim}Fx)$.
- The LSL equivalences + the general quantifier equivalences yield all.

Chapter 6 — Formal Semantics for LMPL

- Venn diagrams can be useful to help us figure out and visualize the conditions under which some *simple* LMPL sentences are true or false.
- But, this technique only works for sentences that have three predicates or less. If a sentence has four predicates or more, then Venn diagrams become quite difficult to draw or comprehend. [Explain this.]
- Chapter 6 provides us with a *general* semantics for LMPL. This will allow us to understand, more generally, the conditions under which *any* (*closed*!) LMPL sentence will be true or false. [Like truth-tables for LSL.]
- In Chapter 6, we will also see a precise definition of the *semantic consequence relation* ($\vDash$) for our new theory LMPL. This will allow us to determine whether LMPL *arguments* are valid or invalid (in general).
- We begin with some new terminology …

Formal Semantics for LMPL I: Some Terminology

- A **domain** ($\mathcal{D}$) is a nonempty (finite) set of individuals.
- The **reference of an individual constant** τ [Ref(τ)] is the object in the domain $\mathcal{D}$ to which τ refers (*e.g.*, ⌜Ref(τ) = x⌝ abbreviates ⌜τ denotes x⌝).
- The **extension of a predicate P** [Ext(**P**)] is the set of all objects in the domain which satisfy **P** (*e.g.*, if P__ : __ is at the podium, and Ref(b) = Branden, then Ext(P) = $\{b\}$). Note: extensions are always subsets of the domain $\mathcal{D}$.
- The **instances of a** (*closed*!) **quantified sentence** ⌜$(Qv)\phi v$⌝ **in a domain** $\mathcal{D}$ are the sentences one gets by replacing all occurrences of v in ⌜ϕv⌝ with the name of each element of $\mathcal{D}$ (*e.g.*, instances of '$(\forall x)Px$' in $\mathcal{D}$ are 'Pa', 'Pb', …, for each individual in $\mathcal{D}$. ∴ there are $|\mathcal{D}|$ instances of ⌜$(Qv)\phi v$⌝ in $\mathcal{D}$).
- An **interpretation** ($\mathcal{I}$) **of an** (*closed*!) **LMPL sentence** p (**or argument** $\mathscr{A}$) is:
  (*i*) a *domain* $\mathcal{D}$,
  (*ii*) an assignment of *extensions* to any *predicate letters* in p ($\mathscr{A}$),
  (*iii*) an assignment of *references* to any *individual constants* in p ($\mathscr{A}$), and
  (*iv*) an assignment of *truth-values* to any *sentence letters* in p ($\mathscr{A}$).

Formal Semantics for LMPL II: $\top$ and $\bot$ in LMPL

- We're now in a position to give precise *truth-conditions* for each kind of (*closed*!) LMPL sentence (augmenting the truth-table definitions of LSL).
- First, the truth conditions for the (*closed*!) *atomic* sentences of LMPL:
  - An atomic sentence $\mathbf{P}\tau$ is *true* ($\top$) on an interpretation $\mathcal{I}$ if the object referred to by the individual constant τ belongs to the extension of the predicate $\mathbf{P}$ (*i.e.*, if $\tau \in \text{Ext}(\mathbf{P})$). If τ does *not* belong to the extension of the predicate $\mathbf{P}$ — that is, if $\tau \notin \text{Ext}(\mathbf{P})$ — then $\mathbf{P}\tau$ is *false* ($\bot$).
- Next, the truth conditions for the (*closed*!) *quantified* sentences of LMPL:
  - A universal sentence $\ulcorner(\forall v)\phi v\urcorner$ is *true* ($\top$) *in* $\mathcal{I}$ if *all* its instances in $\mathcal{I}$ are true. If some of its instances are false (in $\mathcal{I}$), then $\ulcorner(\forall v)\phi v\urcorner$ is *false* ($\bot$).
  - An existential sentence $\ulcorner(\exists v)\phi v\urcorner$ is *true* ($\top$) *in* $\mathcal{I}$ if *some* of its instances are true in $\mathcal{I}$. If *all* its instances are false (in $\mathcal{I}$), then it's *false* ($\bot$).
- NOTE: the usual *truth-tables* for &, $\vee$, $\rightarrow$, $\leftrightarrow$, $\sim$ are still in force in LMPL!

An Example of an LMPL Interpretation

Matrix Representation: $(\mathcal{I})$

	F	G
α	$+$	$-$
β	$-$	$+$

[Ignoring sentence letters.]

- Greek letters 'α'–'σ' (*viz.*, the objects named by the *constants* 'a'–'s') are placed in the left column, alphabetically. All of the predicates in the interpretation $\mathcal{I}$ are placed across the top row, alphabetically. '$+$' means 'satisfies the predicate', and '$-$' means 'does *not* satisfy the predicate'.
- This matrix says (in addition to $\text{Ref}(a) = \alpha$, and $\text{Ref}(b) = \beta$):
  - (*i*) The *domain* $\mathcal{D}$ of $\mathcal{I}$ consists of the two objects α, β (*i.e.*, $\mathcal{D} = \{\alpha, \beta\}$).
  - (*ii*) The *extension* of 'F' consists of the object α (*i.e.*, $\text{Ext}(F) = \{\alpha\}$), and the *extension* of 'G' consists of the object β (*i.e.*, $\text{Ext}(G) = \{\beta\}$).
- **Quiz**: What are the truth-values — in $\mathcal{I}$ — of the following 4 sentences?

(1) $(\exists x)Fx \,\&\, (\exists x)Gx$, (2) $(\exists x)(Fx \,\&\, Gx)$, (3) $(\forall x)(Fx \vee Gx)$, (4) $(\forall x)Fx \vee (\forall x)Gx$

Validity and Invalidity of LMPL Arguments

- An argument-form $\mathscr{A}$ in LMPL is **valid** iff there is no interpretation in which all of $\mathscr{A}$'s premises are true ($\top$), but $\mathscr{A}$'s conclusion is false ($\bot$).

Example: Consider the following LMPL argument-form:

$$(\mathscr{A}_1) \quad \begin{array}{l} (\exists x)Fx \,\&\, (\exists x)Gx \\ \therefore\ (\exists x)(Fx \,\&\, Gx) \end{array}$$

- We have *already* proven that $\mathscr{A}_1$ is *in*valid! We just showed that — in $\mathcal{I}$ — the only premise [(1)] of $\mathscr{A}_1$ is $\top$, but the conclusion [(2)] of $\mathscr{A}_1$ is $\bot$.
- Interpretation $\mathcal{I}$ can also be used to show that the argument-form:

$$(\mathscr{A}_2) \quad \begin{array}{l} (\forall x)(Fx \vee Gx) \\ \therefore\ (\forall x)Fx \vee (\forall x)Gx \end{array}$$

is invalid. Its premise (3) is $\top$ in $\mathcal{I}$, but its conclusion (4) is $\bot$ in $\mathcal{I}$.

More Practice Working with LMPL Interpretations

- Consider the following LMPL interpretation:

($\mathcal{I}_1$)

	F	G	H	I	J
α	+	+	−	+	−
β	−	−	−	+	+
γ	+	−	−	−	+

- So, $\mathcal{I}_1$ is such that: $\mathcal{D} = \{\alpha, \beta, \gamma\}$, $\text{Ext}(F) = \{\alpha, \gamma\}$, $\text{Ext}(G) = \{\alpha\}$, $\text{Ext}(H) = \varnothing$ ($\varnothing$ is the *null set*), $\text{Ext}(I) = \{\alpha, \beta\}$, and $\text{Ext}(J) = \{\beta, \gamma\}$.
- What are the $\mathcal{I}$-truth-values of the following LMPL sentences?

(5) $\sim Ja$

(6) $Fc \rightarrow Ic$

(7) $(\exists x)(Jx \leftrightarrow Hx)$

(8) $(\forall x)[Jx \rightarrow (Gx \vee Fx)]$

(9) $(\exists x)Gx \rightarrow (\forall y)(Fy \vee Gy)$

(10) $(\exists y)(\forall x)[Gy \,\&\, (Jx \rightarrow (Ix \vee Fx))]$

- These are solved on page 1 of my “Working with LMPL Interpretations”.

Constructing LMPL Interpretations to Prove $\nvDash$ Claims

- The notion of *semantic consequence* ($\vDash$) in LMPL is defined in the usual way. We say that $p_1, \ldots, p_n \vDash q$ in LMPL *iff* there is no LMPL interpretation on which all of $p_1, \ldots, p_n$ are true, but q is false.
- In HW #5, you are asked to prove that $p_1, \ldots, p_n \nvDash q$, for various p's and q's. This means you must *construct* (or, *find*) LMPL interpretations on which $p_1, \ldots, p_n$ are all true, but q is false.
- On page 2 of my "Working with LMPL Interpretations" handout, I have included two problems of this kind. There, I explain in detail *how I arrived at* my interpretations. This is a method you should emulate.
- On your HW's and exams, you will ***not*** need to explain *how you arrived at* your interpretations. But, you *will* need to *demonstrate* that your interpretations *really are counterexamples* (*i.e.*, that they *really are* interpretations on which $p_1, \ldots, p_n$ are all true, but q is false).

How Do We *Prove* $\vDash$ Claims in LMPL?

- In LSL, we had *systematic*, truth-table procedures for proving *both* negative ($\nvDash$) *and* affirmative ($\vDash$) semantical claims.
- The method of constructing LMPL interpretations *is* a general way to establish *negative* ($\nvDash$) LMPL-semantical claims.
- We will *not* be learning any systematic methods for (*directly*) establishing *affirmative* ($\vDash$) LMPL-semantical claims. There *are* such methods, but they are beyond the scope of this course.[a]
- In LMPL, we will rely on *natural deduction proofs* to give us an (*in*direct) method for demonstrating the *validity* of LMPL argument-forms. We'll talk about LMPL natural deductions soon.

[a]If an LMPL argument with k predicate letters is *in*valid, then there exists a *counterexample interpretation* $\mathcal{I}$ whose domain $\mathcal{D}$ has no more than 2^k elements. So, *exhaustive search* over *all* interpretations such that $|\mathcal{D}| \leq 2^k$ *is* a *decision procedure* for LMPL-validity. Note: this means checking $2^{2^k \cdot k}$ matrices. This is too many to check, even for small k. If $k = 2$, then $2^{2^k \cdot k} = 2^8 = 256$. For $k = 3$, this is 16777216! See pages 212–215 of Hunter's *Metalogic* (our 140A text). We discuss this in 140A.

Construction of LMPL Interpretations: Examples

- Here are six sample problems that require you to *construct* (or, *find*) LMPL interpretations that are *counterexamples* to $\vDash$ claims (the first two of these are solved on p. 2 of my handout on constructing LMPL interpretations):

(1) $(\forall x)(Fx \to Gx), (\forall x)(Fx \to Hx) \nvDash (\forall x)(Gx \to Hx)$

(2) $(\exists x)(Fx \,\&\, Gx), (\exists x)(Fx \,\&\, Hx), (\forall x)(Gx \to \sim Hx) \nvDash (\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$

(3) $(\forall x)Fx \leftrightarrow (\forall x)Gx \nvDash (\exists x)(Fx \leftrightarrow Gx)$[a]

(4) $(\forall x)Fx \leftrightarrow A \nvDash (\forall x)(Fx \leftrightarrow A)$[b]

(5) $Fa \to (\exists x)Gx \nvDash (\exists x)Fx \to (\exists x)Gx$[c]

(6) $(\exists x)(\forall y)(Fx \to Gy) \nvDash (\exists y)(\forall x)(Fx \to Gy)$[d]

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[a] One solution: $\mathcal{D} = \{a, b\}$, $\text{Ext}(F) = \{a\}$, $\text{Ext}(G) = \{b\}$.

[b] One solution: $\mathcal{D} = \{a, b\}$, 'A' is $\bot$, $\text{Ext}(F) = \{a\}$.

[c] One solution: $\mathcal{D} = \{a, b\}$, $\text{Ext}(F) = \{b\}$, $\text{Ext}(G) = \varnothing$.

[d] One solution: $\mathcal{D} = \{a, b\}$, $\text{Ext}(F) = \{a\}$, $\text{Ext}(G) = \varnothing$.

Construction of LMPL Interpretations: Example #1

(1) $(\forall x)(Fx \to Gx), (\forall x)(Fx \to Hx) \nvDash (\forall x)(Gx \to Hx)$

- To prove (1), we need to construct (find) an interpretation $\mathcal{I}$ such that:
  - (i) '$(\forall x)(Fx \to Gx)$' is true in $\mathcal{I}$.
  - (ii) '$(\forall x)(Fx \to Hx)$' is true in $\mathcal{I}$.
  - (iii) '$(\forall x)(Gx \to Hx)$' is false in $\mathcal{I}$.
- **Step 1**: We begin — *provisionally* — with the smallest domain $\mathcal{D} = \{a\}$.
- **Step 2**: We make sure that the object a is a *counterexample* to the conclusion '$(\forall x)(Gx \to Hx)$'. That is, we make sure that the *instance* '$Ga \to Ha$' of the conclusion is *false* on $\mathcal{I}$. So, we must have $a \in \text{Ext}(G)$, but $a \notin \text{Ext}(H)$. We can achieve this by: $\text{Ext}(G) = \{a\}$, and $\text{Ext}(H) = \varnothing$.
- **Step 3**: At the same time, we try to make *both* of the premises '$(\forall x)(Fx \to Gx)$' *and* '$(\forall x)(Fx \to Hx)$' *true* on $\mathcal{I}$.

- In this case, we can make both premises true simply by ensuring that $a \notin \text{Ext}(F)$. The simplest way to do this is to stipulate that $\text{Ext}(F) = \varnothing$ — which yields the following interpretation that does the trick:

$\mathcal{I}_{(1)}$:

	F	G	H
a	$-$	$+$	$-$

- We have discovered an interpretation $\mathcal{I}_{(1)}$ on which '$(\forall x)(Fx \rightarrow Gx)$' and '$(\forall x)(Fx \rightarrow Hx)$' are both true, but '$(\forall x)(Gx \rightarrow Hx)$' is false (*demonstrate this*!). Therefore, claim (1) is true.

- When you're asked to prove a claim like (1), you must do *2* things:
  - *Report* an interpretation (like $\mathcal{I}_2$) which serves as a counterexample to the validity of the LMPL argument-form, *and*
  - *Demonstrate* that your interpretation *really is* a counterexample — *i.e.*, *show* that your interpretation makes all the premises true and the conclusion false, using the methods above. You do ***not*** need to explain the process which led to the *discovery* of the interpretation.

Construction of LMPL Interpretations: Example #2

(2) $(\exists x)(Fx \,\&\, Gx)$, $(\exists x)(Fx \,\&\, Hx)$, $(\forall x)(Gx \to {\sim}Hx) \nvDash (\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$

- We need an interpretation $\mathcal{I}$ on which '$(\exists x)(Fx \,\&\, Gx)$', '$(\exists x)(Fx \,\&\, Hx)$', and '$(\forall x)(Gx \to {\sim}Hx)$' are all $\top$, but '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$' is $\bot$.
- **Step 1**: We begin with the smallest possible domain $\mathcal{D} = \{a\}$.
- **Step 2**: We make sure that a is a *counterexample* to the conclusion '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$'. So, we make its *instance* '$Fa \leftrightarrow (Ga \vee Ha)$' $\bot$ on $\mathcal{I}$. One way to do this is: $a \in \text{Ext}(F)$, $a \notin \text{Ext}(G)$, and $a \notin \text{Ext}(H)$. So far, we have the following: $\text{Ext}(F) = \{a\}$, and $\text{Ext}(G) = \text{Ext}(H) = \emptyset$.
- **Step 3**: Now, we must make *all three* of the premises (*i*) '$(\exists x)(Fx \,\&\, Gx)$', (*ii*) '$(\exists x)(Fx \,\&\, Hx)$', and (*iii*) '$(\forall x)(Gx \to {\sim}Hx)$' $\top$ on $\mathcal{I}$. In order to make (*i*) $\top$ on $\mathcal{I}$, we must ensure that there is some object in the domain $\mathcal{D}$ which satisfies *both* 'F' *and* 'G'. But, since a must *not* satisfy both 'F' and 'G', this means we will need to *add another object* b to our domain $\mathcal{D}$.

- This new object b must be such that: $b \in \text{Ext}(F)$, and $b \in \text{Ext}(G)$. Now, we have $\text{Ext}(F) = \{a, b\}$, $\text{Ext}(G) = \{b\}$, and $\text{Ext}(H) = \varnothing$.
- All that remains is to ensure that premises (*ii*) and (*iii*) are also $\top$ on $\mathcal{I}$. In order to make (*ii*) $\top$ on $\mathcal{I}$, we'll need to make sure that there is some object in $\mathcal{D}$ which satisfies *both* 'F' *and* 'H'. We could *try* to make b satisfy *all three* 'F', 'G', *and* 'H'. But, if we were to do this, then premise (*iii*) would become *false* on $\mathcal{I}$, since its *instance* '$Gb \rightarrow \sim Hb$' would then be false on $\mathcal{I}$. Thus, we'll need to *add a third object* c to $\mathcal{D}$ such that: $c \in \text{Ext}(F)$, $c \notin \text{Ext}(G)$, and $c \in \text{Ext}(H)$ — and that does the trick:

$\mathcal{I}_{(2)}$:

	F	G	H
a	+	−	−
b	+	+	−
c	+	−	+

- We have discovered an interpretation $\mathcal{I}_{(2)}$ on which '$(\exists x)(Fx \,\&\, Gx)$', '$(\exists x)(Fx \,\&\, Hx)$', and '$(\forall x)(Gx \rightarrow \sim Hx)$' are all $\top$, but on which '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$' is false (*demonstrate this*!). $\therefore$ claim (2) is true.

Construction of LMPL Interpretations for $\nvDash$: Procedure

1. Begin with smallest domain possible $\mathcal{D} = \{\alpha\}$.
2. Make the conclusion of the $\nvDash$ claim false (for α).
   - That is, make the a-instance of the conclusion false.
3. Try to make all premises of the $\nvDash$ claim true (for α).
   - That is, make the a-instance of each of the premises true.
4. If you succeed, then you're done. Now *report and verify* your matrix.
5. If you fail, then add a new individual β to $\mathcal{D} = \{\alpha, \beta\}$, and continue.
6. Make the conclusion of the $\nvDash$ claim false.
   - If the conclusion is an $\forall$ claim, then it's already false.
   - If it's an $\exists$, then you must make sure its b-instance is also false.
7. Make the premises of the $\nvDash$ claim true.
   - If a premise is an $\forall$ claim, then *all* its instances must be true.
   - If it's an $\exists$ claim, only *one* of its instances needs to be true.
8. If you succeed, you're done. If not, add another (γ) to $\mathcal{D}$. Repeat ...

Using Sentential Reasoning to "Verify" LMPL $\vDash$ Claims

$$(\forall x)(\exists y)(Fx \,\&\, Gy) \;\dashv\vDash\; (\exists y)(\forall x)(Fx \,\&\, Gy)$$

- To see why, think about the truth-conditions for each side:

$$(\forall x)(\exists y)(Fx \,\&\, Gy) \approx (\exists y)(Fa \,\&\, Gy) \,\&\, (\exists y)(Fb \,\&\, Gy) \,\&\, \cdots$$

$$\approx [(Fa \,\&\, Ga) \vee (Fa \,\&\, Gb) \vee \cdots] \,\&\, [(Fb \,\&\, Ga) \vee (Fb \,\&\, Gb) \vee \cdots] \,\&\, \cdots$$

$$\approx [Fa \,\&\, (Ga \vee Gb \vee \cdots)] \,\&\, [Fb \,\&\, (Ga \vee Gb \vee \cdots)] \,\&\, \cdots$$

$$\approx (Fa \,\&\, Fb \,\&\, Fc \,\&\, \cdots) \,\&\, (Ga \vee Gb \vee Gc \vee \cdots)$$

$$(\exists y)(\forall x)(Fx \,\&\, Gy) \approx (\forall x)(Fx \,\&\, Ga) \vee (\forall x)(Fx \,\&\, Gb) \vee \cdots$$

$$\approx [(Fa \,\&\, Ga) \,\&\, (Fb \,\&\, Ga) \,\&\, \cdots] \vee [(Fa \,\&\, Gb) \,\&\, (Fb \,\&\, Gb) \,\&\, \cdots] \vee \cdots$$

$$\approx [Ga \,\&\, (Fa \,\&\, Fb \,\&\, \cdots)] \vee [Gb \,\&\, (Fa \,\&\, Fb \,\&\, \cdots)] \vee \cdots$$

$$\approx (Ga \vee Gb \vee Gc \vee \cdots) \,\&\, (Fa \,\&\, Fb \,\&\, Fc \,\&\, \cdots)$$

- $\therefore$ These two formulas are *equivalent*, since the two red formulas are

$$(Ga \vee Gb \vee \cdots) \,\&\, (Fa \,\&\, Fb \,\&\, \cdots) \approx (Fa \,\&\, Fb \,\&\, \cdots) \,\&\, (Ga \vee Gb \vee \cdots)$$