

## Announcements and Such

- Today's Music: *Big Star*
- I have posted my solutions to HW #4 (with the shortest proofs I know).
- I've posted HW #5, which is due on Thursday @ 4pm.
  - A few LMPL symbolization problems (chapter 5).
  - Mostly, working with LMPL Interpretations (chapter 6).
- ☞ I've posted a new handout entitled "Working with LMPL Interpretations", which I will be going over in class today.
- Today: Chapter 6 — LMPL Semantics
  - Working with given LMPL interpretations.
  - Constructing LMPL interpretations to establish  $\neq$  claims.
  - LMPL-validity is decidable (but infeasible).
  - Next Topic: Natural deduction proofs in LMPL.

## Working with LMPL Interpretations (Handout: Part I)

- Consider the following LMPL interpretation:

	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>	<i>J</i>
$\alpha$	+	+	-	+	-
$\beta$	-	-	-	+	+
$\gamma$	+	-	-	-	+

- So,  $\mathcal{I}_1$  is such that:  $\mathcal{D} = \{\alpha, \beta, \gamma\}$ ,  $\text{Ext}(F) = \{\alpha, \gamma\}$ ,  $\text{Ext}(G) = \{\alpha\}$ ,  $\text{Ext}(H) = \emptyset$  ( $\emptyset$  is the *null set*),  $\text{Ext}(I) = \{\alpha, \beta\}$ , and  $\text{Ext}(J) = \{\beta, \gamma\}$ .
- What are the  $\mathcal{I}$ -truth-values of the following LMPL sentences?
 

(5) $\sim Ja$	(8) $(\forall x)[Jx \rightarrow (Gx \vee Fx)]$
(6) $Fc \rightarrow Ic$	(9) $(\exists x)Gx \rightarrow (\forall y)(Fy \vee Gy)$
(7) $(\exists x)(Jx \leftrightarrow Hx)$	(10) $(\exists y)(\forall x)[Gy \& (Jx \rightarrow (Ix \vee Fx))]$
- These are solved on page 1 of my “Working with LMPL Interpretations”.

## ***Constructing* LMPL Interpretations to Prove $\neq$ Claims**

- The notion of *semantic consequence* ( $\models$ ) in LMPL is defined in the usual way. We say that  $p_1, \dots, p_n \models q$  in LMPL *iff* there is no LMPL interpretation on which all of  $p_1, \dots, p_n$  are true, but  $q$  is false.
- In HW #5, you are asked to prove that  $p_1, \dots, p_n \neq q$ , for various  $p$ 's and  $q$ 's. This means you must *construct* (or, *find*) LMPL interpretations on which  $p_1, \dots, p_n$  are all true, but  $q$  is false.
- On page 2 of my “Working with LMPL Interpretations” handout, I have included two problems of this kind. There, I explain in detail *how I arrived at* my interpretations. This is a method you should emulate.
- On your HW's and exams, you will **not** need to explain *how you arrived at* your interpretations. But, you *will* need to *demonstrate* that your interpretations *really are counterexamples* (i.e., that they *really are* interpretations on which  $p_1, \dots, p_n$  are all true, but  $q$  is false).

## Digression: How Do We *Prove* $\models$ Claims in LMPL?

- In LSL, we had *systematic*, truth-table procedures for proving *both* negative ( $\neq$ ) *and* affirmative ( $\models$ ) semantical claims.
- The method of constructing LMPL interpretations *is* a general way to establish *negative* ( $\neq$ ) LMPL-semantical claims.
- We will *not* be learning any systematic methods for (*directly*) establishing *affirmative* ( $\models$ ) LMPL-semantical claims. There *are* such methods, but they are beyond the scope of this course.<sup>a</sup>
- In LMPL, we will rely on *natural deduction proofs* to give us an (*indirect*) method for demonstrating the *validity* of LMPL argument-forms. We'll talk about LMPL natural deductions soon.

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<sup>a</sup>If an LMPL argument with  $k$  predicate letters is *invalid*, then there exists a *counterexample interpretation*  $\mathcal{I}$  whose domain  $\mathcal{D}$  has  $2^k$  elements. So, *exhaustive search* over *all* interpretations such that  $|\mathcal{D}| = 2^k$  is a *decision procedure* for LMPL-validity. Note: this means checking  $2^{2^k \cdot k}$  interpretations/matrices. This is too many to check, even for small  $k$ . If  $k = 2$ , then  $2^{2^k \cdot k} = 2^8 = 256$ . For  $k = 3$ , this is 16777216! See my handout <http://fitelson.org/12A/LMPLd.pdf> for details and references.

## Construction of LMPL Interpretations: Examples

- Here are six sample problems that require you to *construct* (or, *find*) LMPL interpretations that are *counterexamples* to  $\models$  claims (the first two of these are solved on p. 2 of my handout on constructing LMPL interpretations):

$$(1) (\forall x)(Fx \rightarrow Gx), (\forall x)(Fx \rightarrow Hx) \not\models (\forall x)(Gx \rightarrow Hx)$$

$$(2) (\exists x)(Fx \& Gx), (\exists x)(Fx \& Hx), (\forall x)(Gx \rightarrow \sim Hx) \not\models (\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$$

$$(3) (\forall x)Fx \leftrightarrow (\forall x)Gx \not\models (\exists x)(Fx \leftrightarrow Gx)^a$$

$$(4) (\forall x)Fx \leftrightarrow A \not\models (\forall x)(Fx \leftrightarrow A)^b$$

$$(5) Fa \rightarrow (\exists x)Gx \not\models (\exists x)Fx \rightarrow (\exists x)Gx^c$$

$$(6) (\exists x)(\forall y)(Fx \rightarrow Gy) \not\models (\exists y)(\forall x)(Fx \rightarrow Gy)^d$$

<sup>a</sup>One solution:  $\mathcal{D} = \{a, b\}$ ,  $\text{Ext}(F) = \{a\}$ ,  $\text{Ext}(G) = \{b\}$ .

<sup>b</sup>One solution:  $\mathcal{D} = \{a, b\}$ , 'A' is  $\perp$ ,  $\text{Ext}(F) = \{a\}$ .

<sup>c</sup>One solution:  $\mathcal{D} = \{a, b\}$ ,  $\text{Ext}(F) = \{b\}$ ,  $\text{Ext}(G) = \emptyset$ .

<sup>d</sup>One solution:  $\mathcal{D} = \{a, b\}$ ,  $\text{Ext}(F) = \{a\}$ ,  $\text{Ext}(G) = \emptyset$ .

## Construction of LMPL Interpretations: Example #1

(1)  $(\forall x)(Fx \rightarrow Gx), (\forall x)(Fx \rightarrow Hx) \not\models (\forall x)(Gx \rightarrow Hx)$

- To prove (1), we need to construct (find) an interpretation  $\mathcal{I}$  such that:
  - (i)  $'(\forall x)(Fx \rightarrow Gx)'$  is true in  $\mathcal{I}$ .
  - (ii)  $'(\forall x)(Fx \rightarrow Hx)'$  is true in  $\mathcal{I}$ .
  - (iii)  $'(\forall x)(Gx \rightarrow Hx)'$  is false in  $\mathcal{I}$ .
- **Step 1:** We begin — *provisionally* — with the smallest domain  $\mathcal{D} = \{a\}$ .
- **Step 2:** We make sure that the object  $a$  is a *counterexample* to the conclusion  $'(\forall x)(Gx \rightarrow Hx)'$ . That is, we make sure that the *instance*  $'Ga \rightarrow Ha'$  of the conclusion is *false* on  $\mathcal{I}$ . So, we must have  $a \in \text{Ext}(G)$ , but  $a \notin \text{Ext}(H)$ . We can achieve this by:  $\text{Ext}(G) = \{a\}$ , and  $\text{Ext}(H) = \emptyset$ .
- **Step 3:** At the same time, we try to make *both* of the premises  $'(\forall x)(Fx \rightarrow Gx)'$  and  $'(\forall x)(Fx \rightarrow Hx)'$  true on  $\mathcal{I}$ .

- In this case, we can make both premises true simply by ensuring that  $a \notin \text{Ext}(F)$ . The simplest way to do this is to stipulate that  $\text{Ext}(F) = \emptyset$  — which yields the following interpretation that does the trick:

$$\mathcal{I}_{(1)}: \begin{array}{c|ccc} & F & G & H \\ \hline a & - & + & - \end{array}$$

- We have discovered an interpretation  $\mathcal{I}_{(1)}$  on which ‘ $(\forall x)(Fx \rightarrow Gx)$ ’ and ‘ $(\forall x)(Fx \rightarrow Hx)$ ’ are both true, but ‘ $(\forall x)(Gx \rightarrow Hx)$ ’ is false (*demonstrate this!*). Therefore, claim (1) is true.
- When you’re asked to prove a claim like (1), you must do 2 things:
  - *Report* an interpretation (like  $\mathcal{I}_2$ ) which serves as a counterexample to the validity of the LMPL argument-form, *and*
  - *Demonstrate* that your interpretation *really is* a counterexample — *i.e., show* that your interpretation makes all the premises true and the conclusion false, using the methods above. You do **not** need to explain the process which led to the *discovery* of the interpretation.

## Construction of LMPL Interpretations: Example #2

(2)  $(\exists x)(Fx \ \& \ Gx), (\exists x)(Fx \ \& \ Hx), (\forall x)(Gx \ \rightarrow \ \sim Hx) \not\equiv (\forall x)[Fx \ \leftrightarrow \ (Gx \ \vee \ Hx)]$

- We need an interpretation  $\mathcal{I}$  on which ‘ $(\exists x)(Fx \ \& \ Gx)$ ’, ‘ $(\exists x)(Fx \ \& \ Hx)$ ’, and ‘ $(\forall x)(Gx \ \rightarrow \ \sim Hx)$ ’ are all  $\top$ , but ‘ $(\forall x)[Fx \ \leftrightarrow \ (Gx \ \vee \ Hx)]$ ’ is  $\perp$ .
- **Step 1:** We begin with the smallest possible domain  $\mathcal{D} = \{a\}$ .
- **Step 2:** We make sure that  $a$  is a *counterexample* to the conclusion ‘ $(\forall x)[Fx \ \leftrightarrow \ (Gx \ \vee \ Hx)]$ ’. So, we make its *instance* ‘ $Fa \ \leftrightarrow \ (Ga \ \vee \ Ha)$ ’  $\perp$  on  $\mathcal{I}$ . One way to do this is:  $a \in \text{Ext}(F)$ ,  $a \notin \text{Ext}(G)$ , and  $a \notin \text{Ext}(H)$ . So far, we have the following:  $\text{Ext}(F) = \{a\}$ , and  $\text{Ext}(G) = \text{Ext}(H) = \emptyset$ .
- **Step 3:** Now, we must make *all three* of the premises (i) ‘ $(\exists x)(Fx \ \& \ Gx)$ ’, (ii) ‘ $(\exists x)(Fx \ \& \ Hx)$ ’, and (iii) ‘ $(\forall x)(Gx \ \rightarrow \ \sim Hx)$ ’  $\top$  on  $\mathcal{I}$ . In order to make (i)  $\top$  on  $\mathcal{I}$ , we must ensure that there is some object in the domain  $\mathcal{D}$  which satisfies *both* ‘ $F$ ’ and ‘ $G$ ’. But, since  $a$  must *not* satisfy both ‘ $F$ ’ and ‘ $G$ ’, this means we will need to *add another object*  $b$  to our domain  $\mathcal{D}$ .

- This new object  $b$  must be such that:  $b \in \text{Ext}(F)$ , and  $b \in \text{Ext}(G)$ . Now, we have  $\text{Ext}(F) = \{a, b\}$ ,  $\text{Ext}(G) = \{b\}$ , and  $\text{Ext}(H) = \emptyset$ .
- All that remains is to ensure that premises (ii) and (iii) are also  $\top$  on  $\mathcal{I}$ . In order to make (ii)  $\top$  on  $\mathcal{I}$ , we'll need to make sure that there is some object in  $\mathcal{D}$  which satisfies *both* 'F' and 'H'. We could *try* to make  $b$  satisfy *all three* 'F', 'G', and 'H'. But, if we were to do this, then premise (iii) would become *false* on  $\mathcal{I}$ , since its *instance* ' $Gb \rightarrow \sim Hb$ ' would then be false on  $\mathcal{I}$ . Thus, we'll need to *add a third object*  $c$  to  $\mathcal{D}$  such that:  $c \in \text{Ext}(F)$ ,  $c \notin \text{Ext}(G)$ , and  $c \in \text{Ext}(H)$  — and that does the trick:

		$F$	$G$	$H$
$\mathcal{I}_{(2)}$ :	$a$	+	-	-
	$b$	+	+	-
	$c$	+	-	+

- We have discovered an interpretation  $\mathcal{I}_{(2)}$  on which ' $(\exists x)(Fx \ \& \ Gx)$ ', ' $(\exists x)(Fx \ \& \ Hx)$ ', and ' $(\forall x)(Gx \rightarrow \sim Hx)$ ' are all  $\top$ , but on which ' $(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$ ' is false (*demonstrate this!*).  $\therefore$  claim (2) is true.

## Construction of LMPL Interpretations for $\neq$ : Procedure

1. Begin with smallest domain possible  $\mathcal{D} = \{\alpha\}$ .
2. Make the conclusion of the  $\neq$  claim false (for  $\alpha$ ).
  - That is, make the  $a$ -instance of the conclusion false.
3. Try to make all premises of the  $\neq$  claim true (for  $\alpha$ ).
  - That is, make the  $a$ -instance of each of the premises true.
4. If you succeed, then you're done. Now *report and verify* your matrix.
5. If you fail, then add a new individual  $\beta$  to  $\mathcal{D} = \{\alpha, \beta\}$ , and continue.
6. Make the conclusion of the  $\neq$  claim false.
  - If the conclusion is an  $\forall$  claim, then it's already false.
  - If it's an  $\exists$ , then you must make sure its  $b$ -instance is also false.
7. Make the premises of the  $\neq$  claim true.
  - If a premise is an  $\forall$  claim, then *all* its instances must be true.
  - If it's an  $\exists$  claim, only *one* of its instances needs to be true.
8. If you succeed, you're done. If not, add another ( $\gamma$ ) to  $\mathcal{D}$ . Repeat ...

## Using Sentential Reasoning to “Verify” LMPL $\models$ Claims

$$(\forall x)(\exists y)(Fx \& Gy) \models (\exists y)(\forall x)(Fx \& Gy)$$

- To see why, think about the truth-conditions for each side:

$$\begin{aligned} (\forall x)(\exists y)(Fx \& Gy) &\approx (\exists y)(Fa \& Gy) \& (\exists y)(Fb \& Gy) \& \dots \\ &\approx [(Fa \& Ga) \vee (Fa \& Gb) \vee \dots] \& [(Fb \& Ga) \vee (Fb \& Gb) \vee \dots] \& \dots \\ &\approx [Fa \& (Ga \vee Gb \vee \dots)] \& [Fb \& (Ga \vee Gb \vee \dots)] \& \dots \\ &\approx (Fa \& Fb \& Fc \& \dots) \& (Ga \vee Gb \vee Gc \vee \dots) \end{aligned}$$

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$$\begin{aligned} (\exists y)(\forall x)(Fx \& Gy) &\approx (\forall x)(Fx \& Ga) \vee (\forall x)(Fx \& Gb) \vee \dots \\ &\approx [(Fa \& Ga) \& (Fb \& Ga) \& \dots] \vee [(Fa \& Gb) \& (Fb \& Gb) \& \dots] \vee \dots \\ &\approx [Ga \& (Fa \& Fb \& \dots)] \vee [Gb \& (Fa \& Fb \& \dots)] \vee \dots \\ &\approx (Ga \vee Gb \vee Gc \vee \dots) \& (Fa \& Fb \& Fc \& \dots) \end{aligned}$$

- $\therefore$  These two formulas are *equivalent*, since the two red formulas are  $(Ga \vee Gb \vee \dots) \& (Fa \& Fb \& \dots) \approx (Fa \& Fb \& \dots) \& (Ga \vee Gb \vee \dots)$

## Natural Deduction Proofs in LMPL

- The natural deduction rules for LMPL will *include* the rules for LSL that we already know (*viz.*, Ass., &E, &I,  $\rightarrow$ E,  $\rightarrow$ I,  $\sim$ E,  $\sim$ I, DN,  $\vee$ E,  $\vee$ I, *Df.*).
- Plus, we will be *adding* 4 new rules. We will need both introduction and elimination rules for each of the two quantifiers ( $\exists$ I,  $\exists$ E,  $\forall$ I,  $\forall$ E).
- As in LSL, the system will be *sound and complete* (140A!). That is,  $\vdash$  will apply to the same sequents that  $\models$  does in our semantics for LMPL.
- We begin with the simplest: the introduction rule for  $\exists$  ( $\exists$ I). Intuitively, if we have proved  $\phi\tau$  for some individual constant  $\tau$ , then we may infer that  $\phi$  is true of *something* (*e.g.*, that  $(\exists x)\phi x$ ).
- *E.g.*, if we've proved ' $Pa \ \& \ Qa$ ', we may validly infer ' $(\exists x)(Px \ \& \ Qx)$ '.
- We may also infer ' $(\exists x)(Pa \ \& \ Qx)$ ' and ' $(\exists x)(Px \ \& \ Qa)$ ' from ' $Pa \ \& \ Qa$ '.
- These (and similar) considerations lead us to the  $\exists$ I rule ...

## The Rule of $\exists$ -Introduction

**Rule of  $\exists$ -Introduction:** For any sentence  $\phi\tau$ , if  $\phi\tau$  has been inferred at line  $j$  in a proof, then at line  $k$  we may infer ' $(\exists v)\phi v$ ', labeling the line ' $j \exists I$ ' and writing on its left the numbers that occur on the left of  $j$ .

$$\begin{array}{rcl}
 a_1, \dots, a_n & (j) & \phi\tau \\
 & \vdots & \\
 a_1, \dots, a_n & (k) & (\exists v)\phi v \quad j \exists I
 \end{array}$$

Where ' $(\exists v)\phi v$ ' is obtained syntactically from  $\phi\tau$  by:

- Replacing *one or more occurrences* of  $\tau$  in  $\phi\tau$  by a *single* variable  $v$ .
- Note: the variable  $v$  *must not already occur in* the expression  $\phi\tau$ .  
[This prevents *double-binding*, e.g., ' $(\exists x)(\exists x)(Fx \ \& \ Gx)$ '.]
- And, finally, prefixing the quantifier ' $(\exists v)$ ' in front of the resulting expression (which may now have both ' $v$ 's and ' $\tau$ 's occurring in it).

## The Rule of $\forall$ -Elimination

**Rule of  $\forall$ -Elimination:** For any sentence ' $(\forall v)\phi v$ ' and constant  $\tau$ , if ' $(\forall v)\phi v$ ' has been inferred at a line  $j$ , then at line  $k$  we may infer  $\phi\tau$ , labeling the line ' $j \forall E$ ' and writing on its left the numbers that appear on the left of  $j$ .

$$\begin{array}{rcl}
 a_1, \dots, a_n & (j) & (\forall v)\phi v \\
 & \vdots & \\
 a_1, \dots, a_n & (k) & \phi\tau \qquad j \forall E
 \end{array}$$

Where  $\phi\tau$  is obtained syntactically from ' $(\forall v)\phi v$ ' by:

- Deleting the quantifier prefix ' $(\forall v)$ '.
- Replacing *every occurrence* of  $v$  in the open sentence  $\phi v$  by *one and the same* constant  $\tau$ . [This prevents *fallacies*, e.g.,  $(\forall x)(Fx \ \& \ Gx) \not\equiv Fa \ \& \ Gb$ .]
- Note: since ' $\forall$ ' means *everything*, there are *no* restrictions on *which* individual constant may be used in an application of  $\forall E$ .

## An Example Proof Involving Both $\exists$ I and $\forall$ E

Let's prove that  $(\forall x)(Fx \rightarrow Gx), Fa \vdash (\exists x)(\sim Gx \rightarrow Hx)$ .

1	(1) $(\forall x)(Fx \rightarrow Gx)$	Premise
2	(2) $Fa$	Premise
3	(3) $\sim Ga$	Assumption
4	(4) $\sim Ha$	Assumption
1	(5) $Fa \rightarrow Ga$	1 $\forall$ E
1,2	(6) $Ga$	5,2 $\rightarrow$ E
1,2,3	(7) $\Delta$	3,6 $\sim$ E
1,2,3	(8) $\sim \sim Ha$	4,7 $\sim$ I
1,2,3	(9) $Ha$	8 DN
1,2	(10) $\sim Ga \rightarrow Ha$	3,9 $\rightarrow$ I
1,2	(11) $(\exists x)(\sim Gx \rightarrow Hx)$	10 $\exists$ I

- This example illustrates a typical pattern in quantificational proofs: quantifiers are removed from the premises using elimination rules, sentential (*viz.*, LSL) rules are applied, and then quantifiers are reintroduced using introduction rules to obtain the conclusion.

## The Rule of $\forall$ -Introduction: Some Background

- It is useful to think of a universal claim ' $(\forall v)\phi v$ ' as a *conjunction* which asserts that the predicate expression  $\phi$  is satisfied by *all objects* in the domain of discourse (*i.e.*, the conjunction ' $\phi a \& (\phi b \& (\phi c \& \dots))$ ' is true).
- So, in order to be able to *introduce* the universal quantifier (*i.e.*, to *legitimately infer* ' $(\forall v)\phi v$ ' in a proof), we must be in a position to prove  $\phi\tau$ , for *any* individual constant  $\tau$ . This is called *generalizable reasoning*.
- Consider the following *legitimate* introduction of a universal claim:

Problem is:  $(\forall x)(Fx \rightarrow Gx)$ ,  $(\forall x)Fx \vdash (\forall x)Gx$

1	(1)	$(\forall x)(Fx \rightarrow Gx)$	Premise
2	(2)	$(\forall x)Fx$	Premise
1	(3)	$Fa \rightarrow Ga$	1 $\forall E$
2	(4)	$Fa$	2 $\forall E$
1,2	(5)	$Ga$	3,4 $\rightarrow E$
1,2	(6)	$(\forall x)Gx$	5 $\forall I$

## The Rule of $\forall$ -Introduction: II

- We can legitimately infer ' $(\forall x)Gx$ ' at line 6 of this proof, because our inference to ' $Gb$ ' is *generalizable* — *i.e.*, we could have deduced ' $G\tau$ ', for *any* individual constant  $\tau$  — using *exactly parallel* reasoning.
- However, consider the following *illegitimate* “ $\forall$ -Introduction” step:

1	(1)	$(\forall x)(Fx \rightarrow Gx)$	Premise	
2	(2)	$Fb$	Premise	
1	(3)	$Fb \rightarrow Gb$	1 $\forall E$	
1,2	(4)	$Gb$	2,3 $\rightarrow E$	
1,2	(5)	$(\forall x)Gx$	4 $\forall I$	NO!!

- This is *not* a valid inference, since  $(\forall x)(Fx \rightarrow Gx), Fb \not\equiv (\forall x)Gx$ !
- So, what went wrong? The problem is that the inference to ' $Gb$ ' at (4) is *not* generalizable. We can *not* deduce ' $G\tau$ ' — for *any*  $\tau$  — from the premises ' $(\forall x)(Fx \rightarrow Gx)$ ' and ' $Fb$ '. We can *only* infer ' $Gb$ '.

## The Rule of $\forall$ -Introduction: III

**Rule of  $\forall$ -Introduction:** For any sentence  $\phi\tau$ , if  $\phi\tau$  has been inferred at a line  $j$ , then *provided that  $\tau$  does not occur in any premise or assumption whose line number is on the left at line  $j$* , we may infer ' $(\forall v)\phi v$ ' at line  $k$ , labeling the line ' $j \forall I$ ' and writing on its left the same numbers as occur on the left at line  $j$ .

$$\begin{array}{lcl} a_1, \dots, a_n & (j) & \phi\tau \\ & \vdots & \\ a_1, \dots, a_n & (k) & (\forall v)\phi v \quad j \forall I \end{array}$$

Where ' $(\forall v)\phi v$ ' is obtained by:

- Replacing *every* occurrence of  $\tau$  in  $\phi\tau$  with  $v$  and prefixing ' $(\forall v)$ '.  
[Again, 'every' prevents *fallacies*, e.g.,  $(\forall x)(Fx \rightarrow Gx) \neq (\forall x)(\forall y)(Fx \rightarrow Gy)$ .]
- $\tau$  *does not occur in* any of the formulae  $a_1, \dots, a_n$ . [ensures *generalizability*]
- $v$  *does not occur in*  $\phi\tau$ . [prevents *double-binding*]

## The Rule of $\forall$ -Introduction: Four Examples

- Here are four examples of LMPL sequents involving the three quantifier rules we've learned so far ( $\exists$ I,  $\forall$ E, and  $\forall$ I).

$$(1) (\forall x)(Fx \rightarrow Gx) \vdash (\forall x)Fx \rightarrow (\forall x)Gx$$

$$(2) \sim(\exists x)(Fx \& Gx) \vdash (\forall x)(Fx \rightarrow \sim Gx)$$

$$(3) \sim(\forall x)Fx \vdash (\exists x)\sim Fx$$

$$(4) (\forall x)[Fx \rightarrow (\forall y)Gy] \vdash (\forall x)(\forall y)(Fx \rightarrow Gy)$$

### Proof of (1)

Problem is:  $(\forall x)(Fx \rightarrow Gx) \vdash (\forall x)Fx \rightarrow (\forall x)Gx$

1	(1)	$(\forall x)(Fx \rightarrow Gx)$	Premise
2	(2)	$(\forall x)Fx$	Assumption
1	(3)	$Fa \rightarrow Ga$	1 $\forall E$
2	(4)	$Fa$	2 $\forall E$
1,2	(5)	$Ga$	3,4 $\rightarrow E$
1,2	(6)	$(\forall x)Gx$	5 $\forall I$
1	(7)	$(\forall x)Fx \rightarrow (\forall x)Gx$	2,6 $\rightarrow I$

## Proof of (2)

Problem is:  $\sim(\exists x)(Fx \& Gx) \vdash (\forall x)(Fx \rightarrow \sim Gx)$

1	(1)	$\sim(\exists x)(Fx \& Gx)$	Premise
2	(2)	$Fa$	Assumption
3	(3)	$Ga$	Assumption
2,3	(4)	$Fa \& Ga$	2,3 &I
2,3	(5)	$(\exists x)(Fx \& Gx)$	4 $\exists$ I
1,2,3	(6)	$\Delta$	1,5 $\sim$ E
1,2	(7)	$\sim Ga$	3,6 $\sim$ I
1	(8)	$Fa \rightarrow \sim Ga$	2,7 $\rightarrow$ I
1	(9)	$(\forall x)(Fx \rightarrow \sim Gx)$	8 $\forall$ I

### Proof of (3)

Problem is:  $\sim(\forall x)Fx \vdash (\exists x)\sim Fx$

1	(1)	$\sim(\forall x)Fx$	Premise
2	(2)	$\sim(\exists x)\sim Fx$	Assumption
3	(3)	$\sim Fa$	Assumption
3	(4)	$(\exists x)\sim Fx$	3 $\exists I$
2,3	(5)	$\Delta$	2,4 $\sim E$
2	(6)	$\sim\sim Fa$	3,5 $\sim I$
2	(7)	$Fa$	6 DN
2	(8)	$(\forall x)Fx$	7 $\forall I$
1,2	(9)	$\Delta$	1,8 $\sim E$
1	(10)	$\sim\sim(\exists x)\sim Fx$	2,9 $\sim I$
1	(11)	$(\exists x)\sim Fx$	10 DN

### Proof of (4)

Problem is:  $(\forall x)(Fx \rightarrow (\forall y)Gy) \vdash (\forall x)(\forall y)(Fx \rightarrow Gy)$

1	(1)	$(\forall x)(Fx \rightarrow (\forall y)Gy)$	Premise
2	(2)	$Fa$	Assumption
1	(3)	$Fa \rightarrow (\forall y)Gy$	1 $\forall E$
1,2	(4)	$(\forall y)Gy$	3,2 $\rightarrow E$
1,2	(5)	$Gb$	4 $\forall E$
1	(6)	$Fa \rightarrow Gb$	2,5 $\rightarrow I$
1	(7)	$(\forall y)(Fa \rightarrow Gy)$	6 $\forall I$
1	(8)	$(\forall x)(\forall y)(Fx \rightarrow Gy)$	7 $\forall I$

## The Rule of $\exists$ -Elimination: Some Background

- It is useful to think of an existential claim ' $(\exists v)\phi v$ ' as a *disjunction* which asserts that the predicate expression  $\phi$  is satisfied by *at least one* object in the domain (*i.e.*, that the disjunction ' $\phi a \vee (\phi b \vee (\phi c \vee \dots))$ ' is true).
- In this way, we would expect the elimination rule for  $\exists$  to be similar to the elimination rule for  $\vee$ . That is, we'd expect the  $\exists E$  rule to be similar to the  $\vee E$  rule. Indeed, this is the case. It's best to start with a simple example.
- Consider the following *legitimate* elimination of an existential claim:

Problem is:  $(\exists x)(Fx \& Gx) \vdash (\exists x)Fx$

1	(1)	$(\exists x)(Fx \& Gx)$	Premise
2	(2)	$Fa \& Ga$	Assumption
2	(3)	$Fa$	2 &E
2	(4)	$(\exists x)Fx$	3 $\exists I$
1	(5)	$(\exists x)Fx$	1,2,4 $\exists E$

## The Rule of $\exists$ -Elimination: II

- To derive a sentence  $\mathcal{P}$  using the  $\exists E$  rule (with some existential sentence  $\lceil (\exists v)\phi v \rceil$ ), we must first *assume* an *instance*  $\phi\tau$  of  $\lceil (\exists v)\phi v \rceil$ .
- If we can deduce  $\mathcal{P}$  from this assumed instance  $\phi\tau$  — *using generalizable reasoning* — then we may infer  $\mathcal{P}$  *outright*.
- It is because our reasoning from the *instance*  $\phi\tau$  of  $\lceil (\exists v)\phi v \rceil$  to  $\mathcal{P}$  *does not depend on our choice of constant*  $\tau$  (i.e., that our reasoning from  $\phi\tau$  to  $\mathcal{P}$  is *generalizable*) that makes this inference valid.
- When our reasoning is generalizable in this sense, it's as if we are showing that  $\mathcal{P}$  can be deduced from *any* instance  $\phi\tau$  of  $\lceil (\exists v)\phi v \rceil$ .
- As such, this is just like showing that  $\mathcal{P}$  can be deduced from *any disjunct* of the disjunction  $\lceil \phi a \vee (\phi b \vee (\phi c \vee \dots)) \rceil$ . And, this is just like  $\vee E$  reasoning (except that  $\exists E$  only requires *one* assumption).

## The Rule of $\exists$ -Elimination: III

- Here's an *illegitimate* " $\exists$ -Elimination" step:

1	(1)	$(\exists x)Fx$	Premise	
2	(2)	$Ga$	Premise	
3	(3)	$Fa$	Assumption	
2,3	(4)	$Fa \& Ga$	2,3 &I	
2,3	(5)	$(\exists x)(Fx \& Gx)$	4 $\exists$ I	
1,2	(6)	$(\exists x)(Fx \& Gx)$	1,3,5 $\exists$ E	NO!!

- This is *not* a valid inference:  $(\exists x)Fx, Ga \not\equiv (\exists x)(Fx \& Gx)$ !
- So, what went wrong here? The problem is that the inference to ' $(\exists x)(Fx \& Gx)$ ' at line (5) does *not* use *generalizable* reasoning.
- We can *not* legitimately infer ' $(\exists x)(Fx \& Gx)$ ' at line (5) from an *arbitrary instance* ' $F\tau$ ' of ' $(\exists x)Fx$ '. We *must* assume ' $Fa$ ' in *particular* at line (3) in order to deduce ' $(\exists x)(Fx \& Gx)$ ' at line (5).

## The Rule of $\exists$ -Elimination: Official Definition

**$\exists$ -Elimination:** If  $\lceil (\exists v)\phi v \rceil$  occurs at  $i$  depending on  $a_1, \dots, a_n$ , an instance  $\phi\tau$  of  $\lceil (\exists v)\phi v \rceil$  is *assumed* at  $j$ , and  $\mathcal{P}$  is inferred at  $k$  depending on  $b_1, \dots, b_u$ , then at line  $m$  we may infer  $\mathcal{P}$ , with label ‘ $i, j, k \exists E$ ’ and dependencies  $\{a_1, \dots, a_n\} \cup \{b_1, \dots, b_u\}/j$ :

$a_1, \dots, a_n$	(i)	$(\exists v)\phi v$	
	⋮		
$j$	(j)	$\phi\tau$	Assumption
	⋮		
$b_1, \dots, b_u$	(k)	$\mathcal{P}$	
	⋮		
$\{a_1, \dots, a_n\} \cup \{b_1, \dots, b_u\}/j$	(m)	$\mathcal{P}$	$i, j, k \exists E$

Provided that ***all four*** of the following conditions are met:

- $\tau$  (in  $\phi\tau$ ) replaces ***every*** occurrence of  $v$  in  $\phi v$ . [avoids fallacies]
- $\tau$  ***does not occur in***  $\lceil (\exists v)\phi v \rceil$ . [generalizability]
- $\tau$  ***does not occur in***  $\mathcal{P}$ . [generalizability]
- $\tau$  ***does not occur in any*** of  $b_1, \dots, b_u$ , except (possibly)  $\phi\tau$  itself. [generalizability]

## The Rule of $\exists$ -Elimination: Nine Examples

- Here are 9 examples of proofs involving all four quantifier rules.

1.  $(\exists x)\sim Fx \vdash \sim(\forall x)Fx$  [p. 200, example 5]
2.  $(\exists x)(Fx \rightarrow A) \vdash (\forall x)Fx \rightarrow A$  [p. 201, example 6]
3.  $(\forall x)(\forall y)(Gy \rightarrow Fx) \vdash (\forall x)[(\exists y)Gy \rightarrow Fx]$  [p. 203, I. # 19  $\Rightarrow$ ]
4.  $(\exists x)[Fx \rightarrow (\forall y)Gy] \vdash (\exists x)(\forall y)(Fx \rightarrow Gy)$  [p. 203, I. # 20  $\Leftarrow$ ]
5.  $A \vee (\exists x)Fx \vdash (\exists x)(A \vee Fx)$  [p. 203, II. # 2  $\Leftarrow$ ]
6.  $(\exists x)(Fx \& \sim Fx) \vdash (\forall x)(Gx \& \sim Gx)$  [p. 203, I. # 12  $\Rightarrow$ ]
7.  $(\forall x)[Fx \rightarrow (\forall y)\sim Fy] \vdash \sim(\exists x)Fx$  [p. 203, I. # 5]
8.  $(\forall x)(\exists y)(Fx \& Gy) \vdash (\exists y)(\forall x)(Fx \& Gy)$  [p. 201, example 7]
9.  $(\exists y)(\forall x)(Fx \& Gy) \vdash (\forall x)(\exists y)(Fx \& Gy)$  [other direction]

## Proof of (1)

Problem is:  $(\exists x)\sim Fx \vdash \sim(\forall x)Fx$

1	(1)	$(\exists x)\sim Fx$	Premise
2	(2)	$(\forall x)Fx$	Assumption
3	(3)	$\sim Fa$	Assumption
2	(4)	$Fa$	2 $\forall E$
2,3	(5)	$\Delta$	3,4 $\sim E$
1,2	(6)	$\Delta$	1,3,5 $\exists E$
1	(7)	$\sim(\forall x)Fx$	2,6 $\sim I$

## Proof of (2)

Problem is:  $(\exists x)(Fx \rightarrow A) \vdash (\forall x)Fx \rightarrow A$

1	(1) $(\exists x)(Fx \rightarrow A)$	Premise
2	(2) $(\forall x)Fx$	Assumption
3	(3) $Fa \rightarrow A$	Assumption
2	(4) $Fa$	2 $\forall E$
2,3	(5) $A$	3,4 $\rightarrow E$
1,2	(6) $A$	1,3,5 $\exists E$
1	(7) $(\forall x)Fx \rightarrow A$	2,6 $\rightarrow I$

### Proof of (3)

Problem is:  $(\forall x)(\forall y)(Gy \rightarrow Fx) \vdash (\forall x)((\exists y)Gy \rightarrow Fx)$

1	(1) $(\forall x)(\forall y)(Gy \rightarrow Fx)$	Premise
2	(2) $(\exists y)Gy$	Assumption
3	(3) $Gb$	Assumption
1	(4) $(\forall y)(Gy \rightarrow Fa)$	1 $\forall E$
1	(5) $Gb \rightarrow Fa$	4 $\forall E$
1,3	(6) $Fa$	5,3 $\rightarrow E$
1,2	(7) $Fa$	2,3,6 $\exists E$
1	(8) $(\exists y)Gy \rightarrow Fa$	2,7 $\rightarrow I$
1	(9) $(\forall x)((\exists y)Gy \rightarrow Fx)$	8 $\forall I$

## Proof of (4)

Problem is:  $(\exists x)(Fx \rightarrow (\forall y)Gy) \vdash (\exists x)(\forall y)(Fx \rightarrow Gy)$

1	(1) $(\exists x)(Fx \rightarrow (\forall y)Gy)$	Premise
2	(2) $Fa \rightarrow (\forall y)Gy$	Assumption
3	(3) $Fa$	Assumption
2,3	(4) $(\forall y)Gy$	2,3 $\rightarrow E$
2,3	(5) $Gb$	4 $\forall E$
2	(6) $Fa \rightarrow Gb$	3,5 $\rightarrow I$
2	(7) $(\forall y)(Fa \rightarrow Gy)$	6 $\forall I$
2	(8) $(\exists x)(\forall y)(Fx \rightarrow Gy)$	7 $\exists I$
1	(9) $(\exists x)(\forall y)(Fx \rightarrow Gy)$	1,2,8 $\exists E$

## Proof of (5)

Problem is:  $A \vee (\exists x)Fx \vdash (\exists x)(A \vee Fx)$

1	(1) $A \vee (\exists x)Fx$	Premise
2	(2) $A$	Assumption
2	(3) $A \vee Fa$	2 $\vee I$
2	(4) $(\exists x)(A \vee Fx)$	3 $\exists I$
5	(5) $(\exists x)Fx$	Assumption
6	(6) $Fa$	Assumption
6	(7) $A \vee Fa$	6 $\vee I$
6	(8) $(\exists x)(A \vee Fx)$	7 $\exists I$
5	(9) $(\exists x)(A \vee Fx)$	5,6,8 $\exists E$
1	(10) $(\exists x)(A \vee Fx)$	1,2,4,5,9 $\vee E$

## Proof of (6)

Problem is:  $(\exists x)(Fx \& \sim Fx) \vdash (\forall x)(Gx \& \sim Gx)$

1	(1)	$(\exists x)(Fx \& \sim Fx)$	Premise
2	(2)	$Fa \& \sim Fa$	Assumption
3	(3)	$\sim Gb$	Assumption
2	(4)	$\sim Fa$	2 &E
2	(5)	$Fa$	2 &E
2	(6)	$\Delta$	4,5 $\sim$ E
2	(7)	$\sim \sim Gb$	3,6 $\sim$ I
2	(8)	$Gb$	7 DN
9	(9)	$Gb$	Assumption
2	(10)	$\sim Gb$	9,6 $\sim$ I
2	(11)	$Gb \& \sim Gb$	8,10 &I
2	(12)	$(\forall x)(Gx \& \sim Gx)$	11 $\forall$ I
1	(13)	$(\forall x)(Gx \& \sim Gx)$	1,2,12 $\exists$ E

## Proof of (7)

Problem is:  $(\forall x)(Fx \rightarrow (\forall y)\sim Fy) \vdash \sim(\exists x)Fx$

1	(1) $(\forall x)(Fx \rightarrow (\forall y)\sim Fy)$	Premise
2	(2) $(\exists x)Fx$	Assumption
3	(3) $Fa$	Assumption
1	(4) $Fa \rightarrow (\forall y)\sim Fy$	1 $\forall E$
1,3	(5) $(\forall y)\sim Fy$	4,3 $\rightarrow E$
1,3	(6) $\sim Fa$	5 $\forall E$
1,3	(7) $\Delta$	6,3 $\sim E$
1,2	(8) $\Delta$	2,3,7 $\exists E$
1	(9) $\sim(\exists x)Fx$	2,8 $\sim I$

## Proof of (8)

Problem is:  $(\forall x)(\exists y)(Fx \& Gy) \vdash (\exists y)(\forall x)(Fx \& Gy)$

1	(1)	$(\forall x)(\exists y)(Fx \& Gy)$	Premise
1	(2)	$(\exists y)(Fa \& Gy)$	1 $\forall E$
3	(3)	$Fa \& Gb$	Assumption
1	(4)	$(\exists y)(Fc \& Gy)$	1 $\forall E$
5	(5)	$Fc \& Gd$	Assumption
5	(6)	$Fc$	5 $\&E$
1	(7)	$Fc$	4,5,6 $\exists E$
3	(8)	$Gb$	3 $\&E$
1,3	(9)	$Fc \& Gb$	7,8 $\&I$
1,3	(10)	$(\forall x)(Fx \& Gb)$	9 $\forall I$
1,3	(11)	$(\exists y)(\forall x)(Fx \& Gy)$	10 $\exists I$
1	(12)	$(\exists y)(\forall x)(Fx \& Gy)$	2,3,11 $\exists E$

## Proof of (9)

Problem is:  $(\exists y)(\forall x)(Fx \& Gy) \vdash (\forall x)(\exists y)(Fx \& Gy)$

1	(1) $(\exists y)(\forall x)(Fx \& Gy)$	Premise
2	(2) $(\forall x)(Fx \& Gb)$	Assumption
2	(3) $Fa \& Gb$	2 $\forall E$
2	(4) $(\exists y)(Fa \& Gy)$	3 $\exists I$
1	(5) $(\exists y)(Fa \& Gy)$	1,2,4 $\exists E$
1	(6) $(\forall x)(\exists y)(Fx \& Gy)$	5 $\forall I$