Announcements and Such

- Today's Music: *Big Star*
- I have posted my solutions to HW #4 (with the shortest proofs I know).
- I've posted HW #5, which is due on Thursday @ 4pm.
  - A few LMPL symbolization problems (chapter 5).
  - Mostly, working with LMPL Interpretations (chapter 6).

☞ I've posted a new handout entitled "Working with LMPL Interpretations", which I will be going over in class today.

- Today: Chapter 6 — LMPL Semantics
  - Working with given LMPL interpretations.
  - Constructing LMPL interpretations to establish $\nvDash$ claims.
  - LMPL-validity is decidable (but infeasible).
  - Next Topic: Natural deduction proofs in LMPL.

Working with LMPL Interpretations (Handout: Part I)

- Consider the following LMPL interpretation:

		F	G	H	I	J
$(\mathcal{I}_1)$	α	+	+	−	+	−
	β	−	−	−	+	+
	γ	+	−	−	−	+

- So, $\mathcal{I}_1$ is such that: $\mathcal{D} = \{\alpha, \beta, \gamma\}$, $\text{Ext}(F) = \{\alpha, \gamma\}$, $\text{Ext}(G) = \{\alpha\}$, $\text{Ext}(H) = \varnothing$ ($\varnothing$ is the *null set*), $\text{Ext}(I) = \{\alpha, \beta\}$, and $\text{Ext}(J) = \{\beta, \gamma\}$.
- What are the $\mathcal{I}$-truth-values of the following LMPL sentences?

(5) $\sim Ja$

(6) $Fc \rightarrow Ic$

(7) $(\exists x)(Jx \leftrightarrow Hx)$

(8) $(\forall x)[Jx \rightarrow (Gx \vee Fx)]$

(9) $(\exists x)Gx \rightarrow (\forall y)(Fy \vee Gy)$

(10) $(\exists y)(\forall x)[Gy \,\&\, (Jx \rightarrow (Ix \vee Fx))]$

- These are solved on page 1 of my "Working with LMPL Interpretations".

Constructing LMPL Interpretations to Prove $\nvDash$ Claims

- The notion of *semantic consequence* ($\vDash$) in LMPL is defined in the usual way. We say that $p_1, \ldots, p_n \vDash q$ in LMPL *iff* there is no LMPL interpretation on which all of $p_1, \ldots, p_n$ are true, but q is false.
- In HW #5, you are asked to prove that $p_1, \ldots, p_n \nvDash q$, for various p's and q's. This means you must *construct* (or, *find*) LMPL interpretations on which $p_1, \ldots, p_n$ are all true, but q is false.
- On page 2 of my "Working with LMPL Interpretations" handout, I have included two problems of this kind. There, I explain in detail *how I arrived at* my interpretations. This is a method you should emulate.
- On your HW's and exams, you will ***not*** need to explain *how you arrived at* your interpretations. But, you *will* need to *demonstrate* that your interpretations *really are counterexamples* (*i.e.*, that they *really are* interpretations on which $p_1, \ldots, p_n$ are all true, but q is false).

Digression: How Do We *Prove* $\vDash$ Claims in LMPL?

- In LSL, we had *systematic*, truth-table procedures for proving *both* negative ($\nvDash$) *and* affirmative ($\vDash$) semantical claims.
- The method of constructing LMPL interpretations *is* a general way to establish *negative* ($\nvDash$) LMPL-semantical claims.
- We will *not* be learning any systematic methods for (*directly*) establishing *affirmative* ($\vDash$) LMPL-semantical claims. There *are* such methods, but they are beyond the scope of this course.[a]
- In LMPL, we will rely on *natural deduction proofs* to give us an (*in*direct) method for demonstrating the *validity* of LMPL argument-forms. We'll talk about LMPL natural deductions soon.

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[a] If an LMPL argument with k predicate letters is *in*valid, then there exists a *counterexample interpretation* $\mathcal{I}$ whose domain $\mathcal{D}$ has 2^k elements. So, *exhaustive search* over *all* interpretations such that $|\mathcal{D}| = 2^k$ *is* a *decision procedure* for LMPL-validity. Note: this means checking $2^{2^k \cdot k}$ interpretations/matrices. This is too many to check, even for small k. If $k = 2$, then $2^{2^k \cdot k} = 2^8 = 256$. For $k = 3$, this is 16777216! See my handout `http://fitelson.org/12A/LMPLd.pdf` for details and references.

Construction of LMPL Interpretations: Examples

- Here are six sample problems that require you to *construct* (or, *find*) LMPL interpretations that are *counterexamples* to $\vDash$ claims (the first two of these are solved on p. 2 of my handout on constructing LMPL interpretations):

(1) $(\forall x)(Fx \to Gx), (\forall x)(Fx \to Hx) \nvDash (\forall x)(Gx \to Hx)$

(2) $(\exists x)(Fx \,\&\, Gx), (\exists x)(Fx \,\&\, Hx), (\forall x)(Gx \to {\sim}Hx) \nvDash (\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$

(3) $(\forall x)Fx \leftrightarrow (\forall x)Gx \nvDash (\exists x)(Fx \leftrightarrow Gx)$[a]

(4) $(\forall x)Fx \leftrightarrow A \nvDash (\forall x)(Fx \leftrightarrow A)$[b]

(5) $Fa \to (\exists x)Gx \nvDash (\exists x)Fx \to (\exists x)Gx$[c]

(6) $(\exists x)(\forall y)(Fx \to Gy) \nvDash (\exists y)(\forall x)(Fx \to Gy)$[d]

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[a] One solution: $\mathcal{D} = \{a, b\}$, $\text{Ext}(F) = \{a\}$, $\text{Ext}(G) = \{b\}$.

[b] One solution: $\mathcal{D} = \{a, b\}$, 'A' is $\bot$, $\text{Ext}(F) = \{a\}$.

[c] One solution: $\mathcal{D} = \{a, b\}$, $\text{Ext}(F) = \{b\}$, $\text{Ext}(G) = \varnothing$.

[d] One solution: $\mathcal{D} = \{a, b\}$, $\text{Ext}(F) = \{a\}$, $\text{Ext}(G) = \varnothing$.

Construction of LMPL Interpretations: Example #1

(1) $(\forall x)(Fx \to Gx), (\forall x)(Fx \to Hx) \nvDash (\forall x)(Gx \to Hx)$

- To prove (1), we need to construct (find) an interpretation $\mathcal{I}$ such that:
  (i) '$(\forall x)(Fx \to Gx)$' is true in $\mathcal{I}$.
  (ii) '$(\forall x)(Fx \to Hx)$' is true in $\mathcal{I}$.
  (iii) '$(\forall x)(Gx \to Hx)$' is false in $\mathcal{I}$.
- **Step 1**: We begin — *provisionally* — with the smallest domain $\mathcal{D} = \{a\}$.
- **Step 2**: We make sure that the object a is a *counterexample* to the conclusion '$(\forall x)(Gx \to Hx)$'. That is, we make sure that the *instance* '$Ga \to Ha$' of the conclusion is *false* on $\mathcal{I}$. So, we must have $a \in \text{Ext}(G)$, but $a \notin \text{Ext}(H)$. We can achieve this by: $\text{Ext}(G) = \{a\}$, and $\text{Ext}(H) = \varnothing$.
- **Step 3**: At the same time, we try to make *both* of the premises '$(\forall x)(Fx \to Gx)$' *and* '$(\forall x)(Fx \to Hx)$' *true* on $\mathcal{I}$.

- In this case, we can make both premises true simply by ensuring that $a \notin \text{Ext}(F)$. The simplest way to do this is to stipulate that $\text{Ext}(F) = \varnothing$ — which yields the following interpretation that does the trick:

$\mathcal{I}_{(1)}$:

	F	G	H
a	−	+	−

- We have discovered an interpretation $\mathcal{I}_{(1)}$ on which '$(\forall x)(Fx \to Gx)$' and '$(\forall x)(Fx \to Hx)$' are both true, but '$(\forall x)(Gx \to Hx)$' is false (*demonstrate this*!). Therefore, claim (1) is true.
- When you're asked to prove a claim like (1), you must do *2* things:
  - *Report* an interpretation (like $\mathcal{I}_2$) which serves as a counterexample to the validity of the LMPL argument-form, *and*
  - *Demonstrate* that your interpretation *really is* a counterexample — *i.e.*, *show* that your interpretation makes all the premises true and the conclusion false, using the methods above. You do ***not*** need to explain the process which led to the *discovery* of the interpretation.

Construction of LMPL Interpretations: Example #2

(2) $(\exists x)(Fx \,\&\, Gx), (\exists x)(Fx \,\&\, Hx), (\forall x)(Gx \to {\sim}Hx) \nvDash (\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$

- We need an interpretation $\mathcal{I}$ on which '$(\exists x)(Fx \,\&\, Gx)$', '$(\exists x)(Fx \,\&\, Hx)$', and '$(\forall x)(Gx \to {\sim}Hx)$' are all $\top$, but '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$' is $\bot$.
- **Step 1**: We begin with the smallest possible domain $\mathcal{D} = \{a\}$.
- **Step 2**: We make sure that a is a *counterexample* to the conclusion '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$'. So, we make its *instance* '$Fa \leftrightarrow (Ga \vee Ha)$' $\bot$ on $\mathcal{I}$. One way to do this is: $a \in \text{Ext}(F)$, $a \notin \text{Ext}(G)$, and $a \notin \text{Ext}(H)$. So far, we have the following: $\text{Ext}(F) = \{a\}$, and $\text{Ext}(G) = \text{Ext}(H) = \varnothing$.
- **Step 3**: Now, we must make *all three* of the premises (*i*) '$(\exists x)(Fx \,\&\, Gx)$', (*ii*) '$(\exists x)(Fx \,\&\, Hx)$', and (*iii*) '$(\forall x)(Gx \to {\sim}Hx)$' $\top$ on $\mathcal{I}$. In order to make (*i*) $\top$ on $\mathcal{I}$, we must ensure that there is some object in the domain $\mathcal{D}$ which satisfies *both* 'F' *and* 'G'. But, since a must *not* satisfy both 'F' and 'G', this means we will need to *add another object* b to our domain $\mathcal{D}$.

- This new object b must be such that: $b \in \text{Ext}(F)$, and $b \in \text{Ext}(G)$. Now, we have $\text{Ext}(F) = \{a, b\}$, $\text{Ext}(G) = \{b\}$, and $\text{Ext}(H) = \varnothing$.
- All that remains is to ensure that premises (*ii*) and (*iii*) are also $\top$ on $\mathcal{I}$. In order to make (*ii*) $\top$ on $\mathcal{I}$, we'll need to make sure that there is some object in $\mathcal{D}$ which satisfies *both* 'F' *and* 'H'. We could *try* to make b satisfy *all three* 'F', 'G', *and* 'H'. But, if we were to do this, then premise (*iii*) would become *false* on $\mathcal{I}$, since its *instance* '$Gb \to {\sim}Hb$' would then be false on $\mathcal{I}$. Thus, we'll need to *add a third object* c to $\mathcal{D}$ such that: $c \in \text{Ext}(F)$, $c \notin \text{Ext}(G)$, and $c \in \text{Ext}(H)$ — and that does the trick:

$\mathcal{I}_{(2)}$:

	F	G	H
a	$+$	$-$	$-$
b	$+$	$+$	$-$
c	$+$	$-$	$+$

- We have discovered an interpretation $\mathcal{I}_{(2)}$ on which '$(\exists x)(Fx \,\&\, Gx)$', '$(\exists x)(Fx \,\&\, Hx)$', and '$(\forall x)(Gx \to {\sim}Hx)$' are all $\top$, but on which '$(\forall x)[Fx \leftrightarrow (Gx \vee Hx)]$' is false (*demonstrate this*!). $\therefore$ claim (2) is true.

Construction of LMPL Interpretations for $\nvDash$: Procedure

1. Begin with smallest domain possible $\mathcal{D} = \{\alpha\}$.
2. Make the conclusion of the $\nvDash$ claim false (for α).
   - That is, make the a-instance of the conclusion false.
3. Try to make all premises of the $\nvDash$ claim true (for α).
   - That is, make the a-instance of each of the premises true.
4. If you succeed, then you're done. Now *report and verify* your matrix.
5. If you fail, then add a new individual β to $\mathcal{D} = \{\alpha, \beta\}$, and continue.
6. Make the conclusion of the $\nvDash$ claim false.
   - If the conclusion is an $\forall$ claim, then it's already false.
   - If it's an $\exists$, then you must make sure its b-instance is also false.
7. Make the premises of the $\nvDash$ claim true.
   - If a premise is an $\forall$ claim, then *all* its instances must be true.
   - If it's an $\exists$ claim, only *one* of its instances needs to be true.
8. If you succeed, you're done. If not, add another (γ) to $\mathcal{D}$. Repeat …

Using Sentential Reasoning to "Verify" LMPL $\vDash$ Claims

$$(\forall x)(\exists y)(Fx \,\&\, Gy) \;{=}\!\!\vDash\; (\exists y)(\forall x)(Fx \,\&\, Gy)$$

- To see why, think about the truth-conditions for each side:

$$(\forall x)(\exists y)(Fx \,\&\, Gy) \approx (\exists y)(Fa \,\&\, Gy) \,\&\, (\exists y)(Fb \,\&\, Gy) \,\&\, \cdots$$

$$\approx [(Fa \,\&\, Ga) \vee (Fa \,\&\, Gb) \vee \cdots] \,\&\, [(Fb \,\&\, Ga) \vee (Fb \,\&\, Gb) \vee \cdots] \,\&\, \cdots$$

$$\approx [Fa \,\&\, (Ga \vee Gb \vee \cdots)] \,\&\, [Fb \,\&\, (Ga \vee Gb \vee \cdots)] \,\&\, \cdots$$

$$\approx (Fa \,\&\, Fb \,\&\, Fc \,\&\, \cdots) \,\&\, (Ga \vee Gb \vee Gc \vee \cdots)$$

$$(\exists y)(\forall x)(Fx \,\&\, Gy) \approx (\forall x)(Fx \,\&\, Ga) \vee (\forall x)(Fx \,\&\, Gb) \vee \cdots$$

$$\approx [(Fa \,\&\, Ga) \,\&\, (Fb \,\&\, Ga) \,\&\, \cdots] \vee [(Fa \,\&\, Gb) \,\&\, (Fb \,\&\, Gb) \,\&\, \cdots] \vee \cdots$$

$$\approx [Ga \,\&\, (Fa \,\&\, Fb \,\&\, \cdots)] \vee [Gb \,\&\, (Fa \,\&\, Fb \,\&\, \cdots)] \vee \cdots$$

$$\approx (Ga \vee Gb \vee Gc \vee \cdots) \,\&\, (Fa \,\&\, Fb \,\&\, Fc \,\&\, \cdots)$$

- $\therefore$ These two formulas are *equivalent*, since the two red formulas are $(Ga \vee Gb \vee \cdots) \,\&\, (Fa \,\&\, Fb \,\&\, \cdots) \approx (Fa \,\&\, Fb \,\&\, \cdots) \,\&\, (Ga \vee Gb \vee \cdots)$

Natural Deduction Proofs in LMPL

- The natural deduction rules for LMPL will *include* the rules for LSL that we already know (*viz.*, Ass., &E, &I, →E, →I, ~E, ~I, DN, ∨E, ∨I, *Df*.).
- Plus, we will be *adding* 4 new rules. We will need both introduction and elimination rules for each of the two quantifiers (∃I, ∃E, ∀I, ∀E).
- As in LSL, the system will be *sound and complete* (140A!). That is, $\vdash$ will apply to the same sequents that $\vDash$ does in our semantics for LMPL.
- We begin with the simplest: the introduction rule for $\exists$ ($\exists$I). Intuitively, if we have proved $\phi\tau$ for some individual constant τ, then we may infer that ϕ is true of *something* (*e.g.*, that $(\exists x)\phi x$).
- *E.g.*, if we've proved '$Pa \,\&\, Qa$', we may validly infer '$(\exists x)(Px \,\&\, Qx)$'.
- We may also infer '$(\exists x)(Pa \,\&\, Qx)$' and '$(\exists x)(Px \,\&\, Qa)$' from '$Pa \,\&\, Qa$'.
- These (and similar) considerations lead us to the $\exists$I rule …

The Rule of ∃-Introduction

Rule of ∃-Introduction: For any sentence $\phi\tau$, if $\phi\tau$ has been inferred at line j in a proof, then at line k we may infer $\ulcorner(\exists\nu)\phi\nu\urcorner$, labeling the line 'j ∃I' and writing on its left the numbers that occur on the left of j.

$$\begin{array}{llll} a_1,\ldots,a_n & (j) & \phi\tau & \\ & \vdots & & \\ a_1,\ldots,a_n & (k) & (\exists\nu)\phi\nu & j\ \exists I \end{array}$$

Where $\ulcorner(\exists\nu)\phi\nu\urcorner$ is obtained syntactically from $\phi\tau$ by:

- Replacing ***one or more occurrences*** of τ in $\phi\tau$ by a *single* variable ν.
- Note: the variable ν ***must not already occur in*** the expression $\phi\tau$. [This prevents *double-binding*, *e.g.*, '$(\exists x)(\exists x)(Fx \,\&\, Gx)$'.]
- And, finally, prefixing the quantifier $\ulcorner(\exists\nu)\urcorner$ in front of the resulting expression (which may now have both $\ulcorner\nu\urcorner$s and $\ulcorner\tau\urcorner$s occurring in it).

The Rule of ∀-Elimination

Rule of ∀-Elimination: For any sentence $\ulcorner(\forall\nu)\phi\nu\urcorner$ and constant τ, if $\ulcorner(\forall\nu)\phi\nu\urcorner$ has been inferred at a line j, then at line k we may infer $\phi\tau$, labeling the line 'j ∀E' and writing on its left the numbers that appear on the left of j.

$$\begin{array}{llll} a_1,\ldots,a_n & (j) & (\forall\nu)\phi\nu & \\ & \vdots & & \\ a_1,\ldots,a_n & (k) & \phi\tau & j\ \forall E \end{array}$$

Where $\phi\tau$ is obtained syntactically from $\ulcorner(\forall\nu)\phi\nu\urcorner$ by:

- Deleting the quantifier prefix $\ulcorner(\forall\nu)\urcorner$.
- Replacing ***every occurrence*** of ν in the open sentence $\phi\nu$ by ***one and the same*** constant τ. [This prevents *fallacies*, *e.g.*, $(\forall x)(Fx \,\&\, Gx) \nvdash Fa \,\&\, Gb$.]
- Note: since '∀' means *everything*, there are *no* restrictions on *which* individual constant may be used in an application of ∀E.

An Example Proof Involving Both ∃I and ∀E

Let's prove that $(\forall x)(Fx \to Gx), Fa \vdash (\exists x)(\sim Gx \to Hx)$.

1	(1)	(∀x)(Fx→Gx)	Premise
2	(2)	Fa	Premise
3	(3)	~Ga	Assumption
4	(4)	~Ha	Assumption
1	(5)	Fa→Ga	1 ∀E
1,2	(6)	Ga	5,2 →E
1,2,3	(7)	⋀	3,6 ~E
1,2,3	(8)	~~Ha	4,7 ~I
1,2,3	(9)	Ha	8 DN
1,2	(10)	~Ga→Ha	3,9 →I
1,2	(11)	(∃x)(~Gx→Hx)	10 ∃I

- This example illustrates a typical pattern in quantificational proofs: quantifiers are removed from the premises using elimination rules, sentential (*viz.*, LSL) rules are applied, and then quantifiers are reintroduced using introduction rules to obtain the conclusion.

The Rule of ∀-Introduction: Some Background

- It is useful to think of a universal claim $\ulcorner(\forall\nu)\phi\nu\urcorner$ as a *conjunction* which asserts that the predicate expression ϕ is satisfied by *all objects* in the domain of discourse (*i.e.*, the conjunction $\ulcorner\phi a \,\&\, (\phi b \,\&\, (\phi c \,\&\, \ldots))\urcorner$ is true).
- So, in order to be able to *introduce* the universal quantifier (*i.e.*, to *legitimately infer* $\ulcorner(\forall\nu)\phi\nu\urcorner$ in a proof), we must be in a position to prove $\phi\tau$, for *any* individual constant τ. This is called *generalizable reasoning*.
- Consider the following *legitimate* introduction of a universal claim:

Problem is: (∀x)(Fx→Gx), (∀x)Fx ⊦ (∀x)Gx

1	(1)	(∀x)(Fx→Gx)	Premise
2	(2)	(∀x)Fx	Premise
1	(3)	Fa→Ga	1 ∀E
2	(4)	Fa	2 ∀E
1,2	(5)	Ga	3,4 →E
1,2	(6)	(∀x)Gx	5 ∀I

The Rule of ∀-Introduction: II

- We can legitimately infer '$(\forall x)Gx$' at line 6 of this proof, because our inference to 'Gb' is *generalizable* — *i.e.*, we could have deduced $\ulcorner G\tau \urcorner$, for *any* individual constant τ — using *exactly parallel* reasoning.
- However, consider the following *il*legitimate "∀-Introduction" step:

1	(1)	(∀x)(Fx→Gx)	Premise	
2	(2)	Fb	Premise	
1	(3)	Fb→Gb	1 ∀E	
1,2	(4)	Gb	2,3 →E	
1,2	(5)	(∀x)Gx	4 ∀I	NO!!

- This is *not* a valid inference, since $(\forall x)(Fx \to Gx), Fb \nvDash (\forall x)Gx$!
- So, what went wrong? The problem is that the inference to 'Gb' at (4) is *not* generalizable. We can *not* deduce $\ulcorner G\tau \urcorner$ — for *any* τ — from the premises '$(\forall x)(Fx \to Gx)$' and 'Fb'. We can *only* infer '$G\boldsymbol{b}$'.

The Rule of ∀-Introduction: III

Rule of ∀-Introduction: For any sentence $\phi\tau$, if $\phi\tau$ has been inferred at a line j, then *provided that τ does not occur in any premise or assumption whose line number is on the left at line* j, we may infer $\ulcorner(\forall\nu)\phi\nu\urcorner$ at line k, labeling the line 'j ∀I' and writing on its left the same numbers as occur on the left at line j.

$a_1,\ldots,a_n$	(j)	$\phi\tau$	
	$\vdots$		
$a_1,\ldots,a_n$	(k)	$(\forall\nu)\phi\nu$	j ∀I

Where $\ulcorner(\forall\nu)\phi\nu\urcorner$ is obtained by:

- Replacing ***every*** occurrence of τ in $\phi\tau$ with ν and prefixing $\ulcorner(\forall\nu)\urcorner$.
  [Again, 'every' prevents *fallacies*, *e.g.*, $(\forall x)(Fx \to Gx) \nvdash (\forall x)(\forall y)(Fx \to Gy)$.]
- τ ***does not occur in*** any of the formulae $a_1,\ldots,a_n$. [ensures *generalizability*]
- ν ***does not occur in*** $\phi\tau$. [prevents *double-binding*]

The Rule of ∀-Introduction: Four Examples

- Here are four examples of LMPL sequents involving the three quantifier rules we've learned so far (∃I, ∀E, and ∀I).

(1) $(\forall x)(Fx \to Gx) \vdash (\forall x)Fx \to (\forall x)Gx$

(2) $\sim(\exists x)(Fx \,\&\, Gx) \vdash (\forall x)(Fx \to \sim Gx)$

(3) $\sim(\forall x)Fx \vdash (\exists x)\sim Fx$

(4) $(\forall x)[Fx \to (\forall y)Gy] \vdash (\forall x)(\forall y)(Fx \to Gy)$

Proof of (1)

Problem is: (∀x)(Fx→Gx) ⊢ (∀x)Fx→(∀x)Gx

1	(1)	(∀x)(Fx→Gx)	Premise
2	(2)	(∀x)Fx	Assumption
1	(3)	Fa→Ga	1 ∀E
2	(4)	Fa	2 ∀E
1,2	(5)	Ga	3,4 →E
1,2	(6)	(∀x)Gx	5 ∀I
1	(7)	(∀x)Fx→(∀x)Gx	2,6 →I

Proof of (2)

Problem is: ~(∃x)(Fx&Gx) ⊦ (∀x)(Fx→~Gx)

1	(1)	~(∃x)(Fx&Gx)	Premise
2	(2)	Fa	Assumption
3	(3)	Ga	Assumption
2,3	(4)	Fa&Ga	2,3 &I
2,3	(5)	(∃x)(Fx&Gx)	4 ∃I
1,2,3	(6)	Λ	1,5 ~E
1,2	(7)	~Ga	3,6 ~I
1	(8)	Fa→~Ga	2,7 →I
1	(9)	(∀x)(Fx→~Gx)	8 ∀I

Proof of (3)

Problem is: ~(∀x)Fx ⊦ (∃x)~Fx

1	(1)	~(∀x)Fx	Premise
2	(2)	~(∃x)~Fx	Assumption
3	(3)	~Fa	Assumption
3	(4)	(∃x)~Fx	3 ∃I
2,3	(5)	Λ	2,4 ~E
2	(6)	~~Fa	3,5 ~I
2	(7)	Fa	6 DN
2	(8)	(∀x)Fx	7 ∀I
1,2	(9)	Λ	1,8 ~E
1	(10)	~~(∃x)~Fx	2,9 ~I
1	(11)	(∃x)~Fx	10 DN

Proof of (4)

Problem is: (∀x)(Fx→(∀y)Gy) ⊦ (∀x)(∀y)(Fx→Gy)

1	(1)	(∀x)(Fx→(∀y)Gy)	Premise
2	(2)	Fa	Assumption
1	(3)	Fa→(∀y)Gy	1 ∀E
1,2	(4)	(∀y)Gy	3,2 →E
1,2	(5)	Gb	4 ∀E
1	(6)	Fa→Gb	2,5 →I
1	(7)	(∀y)(Fa→Gy)	6 ∀I
1	(8)	(∀x)(∀y)(Fx→Gy)	7 ∀I

The Rule of ∃-Elimination: Some Background

- It is useful to think of an existential claim $\ulcorner(\exists\nu)\phi\nu\urcorner$ as a *disjunction* which asserts that the predicate expression ϕ is satisfied by *at least one* object in the domain (*i.e.*, that the disjunction $\ulcorner\phi a \vee (\phi b \vee (\phi c \vee \ldots))\urcorner$ is true).
- In this way, we would expect the elimination rule for ∃ to be similar to the elimination rule for ∨. That is, we'd expect the ∃E rule to be similar to the ∨E rule. Indeed, this is the case. It's best to start with a simple example.
- Consider the following *legitimate* elimination of an existential claim:

Problem is: (∃x)(Fx&Gx) ⊦ (∃x)Fx

1	(1)	(∃x)(Fx&Gx)	Premise
2	(2)	Fa&Ga	Assumption
2	(3)	Fa	2 &E
2	(4)	(∃x)Fx	3 ∃I
1	(5)	(∃x)Fx	1,2,4 ∃E

The Rule of ∃-Elimination: II

- To derive a sentence $\mathscr{P}$ using the ∃E rule (with some existential sentence $\ulcorner(\exists\nu)\phi\nu\urcorner$), we must first *assume* an *instance* $\phi\tau$ of $\ulcorner(\exists\nu)\phi\nu\urcorner$.
- If we can deduce $\mathscr{P}$ from this assumed instance $\phi\tau$ — *using **generalizable** reasoning* — then we may infer $\mathscr{P}$ *outright*.
- It is because our reasoning from the *instance* $\phi\tau$ of $\ulcorner(\exists\nu)\phi\nu\urcorner$ to $\mathscr{P}$ *does not depend on our choice of constant* τ (*i.e.*, that our reasoning from $\phi\tau$ to $\mathscr{P}$ is *generalizable*) that makes this inference valid.
- When our reasoning is generalizable in this sense, it's as if we are showing that $\mathscr{P}$ can be deduced from *any* instance $\phi\tau$ of $\ulcorner(\exists\nu)\phi\nu\urcorner$.
- As such, this is just like showing that $\mathscr{P}$ can be deduced from *any disjunct* of the disjunction $\ulcorner\phi a \vee (\phi b \vee (\phi c \vee \ldots))\urcorner$. And, this is just like ∨E reasoning (except that ∃E only requires *one* assumption).

The Rule of ∃-Elimination: III

- Here's an *il*legitimate "∃-Elimination" step:

1	(1)	(∃x)Fx	Premise	
2	(2)	Ga	Premise	
3	(3)	Fa	Assumption	
2,3	(4)	Fa&Ga	2,3 &I	
2,3	(5)	(∃x)(Fx&Gx)	4 ∃I	
1,2	(6)	(∃x)(Fx&Gx)	1,3,5 ∃E	NO!!

- This is *not* a valid inference: $(\exists x)Fx, Ga \nvDash (\exists x)(Fx \,\&\, Gx)$!
- So, what went wrong here? The problem is that the inference to '$(\exists x)(Fx \,\&\, Gx)$' at line (5) does *not* use *generalizable* reasoning.
- We can *not* legitimately infer '$(\exists x)(Fx \,\&\, Gx)$' at line (5) from an *arbitrary instance* $\ulcorner F\tau\urcorner$ of '$(\exists x)Fx$'. We *must* assume '$F\boldsymbol{a}$' *in particular* at line (3) in order to deduce '$(\exists x)(Fx \,\&\, Gx)$' at line (5).

The Rule of ∃-Elimination: Official Definition

∃-Elimination: If $\ulcorner(\exists\nu)\phi\nu\urcorner$ occurs at i depending on $a_1,\ldots,a_n$, an instance $\phi\tau$ of $\ulcorner(\exists\nu)\phi\nu\urcorner$ is *assumed* at j, and $\mathscr{P}$ is inferred at k depending on $b_1,\ldots,b_u$, then at line m we may infer $\mathscr{P}$, with label 'i, j, k ∃E' and dependencies $\{a_1,\ldots,a_n\} \cup \{b_1,\ldots,b_u\}$/j:

$a_1,\ldots,a_n$	(i)	$(\exists\nu)\phi\nu$	
	⋮		
j	(j)	$\phi\tau$	Assumption
	⋮		
$b_1,\ldots,b_u$	(k)	$\mathscr{P}$	
	⋮		
$\{a_1,\ldots,a_n\} \cup \{b_1,\ldots,b_u\}$/j	(m)	$\mathscr{P}$	i, j, k ∃E

Provided that ***all four*** of the following conditions are met:

- τ (in $\phi\tau$) replaces ***every*** occurrence of ν in $\phi\nu$. [avoids fallacies]
- τ ***does not occur in*** $\ulcorner(\exists\nu)\phi\nu\urcorner$. [generalizability]
- τ ***does not occur in*** $\mathscr{P}$. [generalizability]
- τ ***does not occur in*** *any* of $b_1,\ldots,b_u$, except (possibly) $\phi\tau$ itself. [generalizability]

The Rule of ∃-Elimination: Nine Examples

- Here are 9 examples of proofs involving all four quantifier rules.

1. $(\exists x){\sim}Fx \vdash {\sim}(\forall x)Fx$ [*p*. 200, example 5]
2. $(\exists x)(Fx \to A) \vdash (\forall x)Fx \to A$ [*p*. 201, example 6]
3. $(\forall x)(\forall y)(Gy \to Fx) \vdash (\forall x)[(\exists y)Gy \to Fx]$ [*p*. 203, I. # 19 ⇒]
4. $(\exists x)[Fx \to (\forall y)Gy] \vdash (\exists x)(\forall y)(Fx \to Gy)$ [*p*. 203, I. # 20 ⇐]
5. $A \vee (\exists x)Fx \vdash (\exists x)(A \vee Fx)$ [*p*. 203, II. # 2 ⇐]
6. $(\exists x)(Fx \,\&\, {\sim}Fx) \vdash (\forall x)(Gx \,\&\, {\sim}Gx)$ [*p*. 203, I. # 12 ⇒]
7. $(\forall x)[Fx \to (\forall y){\sim}Fy] \vdash {\sim}(\exists x)Fx$ [*p*. 203, I. # 5]
8. $(\forall x)(\exists y)(Fx \,\&\, Gy) \vdash (\exists y)(\forall x)(Fx \,\&\, Gy)$ [*p*. 201, example 7]
9. $(\exists y)(\forall x)(Fx \,\&\, Gy) \vdash (\forall x)(\exists y)(Fx \,\&\, Gy)$ [other direction]

Proof of (1)

Problem is: (∃x)~Fx ⊦ ~(∀x)Fx

1	(1)	(∃x)~Fx	Premise
2	(2)	(∀x)Fx	Assumption
3	(3)	~Fa	Assumption
2	(4)	Fa	2 ∀E
2,3	(5)	⋀	3,4 ~E
1,2	(6)	⋀	1,3,5 ∃E
1	(7)	~(∀x)Fx	2,6 ~I

Proof of (2)

Problem is: (∃x)(Fx→A) ⊦ (∀x)Fx→A

1	(1)	(∃x)(Fx→A)	Premise
2	(2)	(∀x)Fx	Assumption
3	(3)	Fa→A	Assumption
2	(4)	Fa	2 ∀E
2,3	(5)	A	3,4 →E
1,2	(6)	A	1,3,5 ∃E
1	(7)	(∀x)Fx→A	2,6 →I

Proof of (3)

Problem is: (∀x)(∀y)(Gy→Fx) ⊦ (∀x)((∃y)Gy→Fx)

1	(1)	(∀x)(∀y)(Gy→Fx)	Premise
2	(2)	(∃y)Gy	Assumption
3	(3)	Gb	Assumption
1	(4)	(∀y)(Gy→Fa)	1 ∀E
1	(5)	Gb→Fa	4 ∀E
1,3	(6)	Fa	5,3 →E
1,2	(7)	Fa	2,3,6 ∃E
1	(8)	(∃y)Gy→Fa	2,7 →I
1	(9)	(∀x)((∃y)Gy→Fx)	8 ∀I

Proof of (4)

Problem is: (∃x)(Fx→(∀y)Gy) ⊦ (∃x)(∀y)(Fx→Gy)

1	(1)	(∃x)(Fx→(∀y)Gy)	Premise
2	(2)	Fa→(∀y)Gy	Assumption
3	(3)	Fa	Assumption
2,3	(4)	(∀y)Gy	2,3 →E
2,3	(5)	Gb	4 ∀E
2	(6)	Fa→Gb	3,5 →I
2	(7)	(∀y)(Fa→Gy)	6 ∀I
2	(8)	(∃x)(∀y)(Fx→Gy)	7 ∃I
1	(9)	(∃x)(∀y)(Fx→Gy)	1,2,8 ∃E

Proof of (5)

Problem is: A∨(∃x)Fx ⊢ (∃x)(A∨Fx)

1	(1)	A∨(∃x)Fx	Premise
2	(2)	A	Assumption
2	(3)	A∨Fa	2 ∨I
2	(4)	(∃x)(A∨Fx)	3 ∃I
5	(5)	(∃x)Fx	Assumption
6	(6)	Fa	Assumption
6	(7)	A∨Fa	6 ∨I
6	(8)	(∃x)(A∨Fx)	7 ∃I
5	(9)	(∃x)(A∨Fx)	5,6,8 ∃E
1	(10)	(∃x)(A∨Fx)	1,2,4,5,9 ∨E

Proof of (6)

Problem is: (∃x)(Fx&~Fx) ⊢ (∀x)(Gx&~Gx)

1	(1)	(∃x)(Fx&~Fx)	Premise
2	(2)	Fa&~Fa	Assumption
3	(3)	~Gb	Assumption
2	(4)	~Fa	2 &E
2	(5)	Fa	2 &E
2	(6)	Λ	4,5 ~E
2	(7)	~~Gb	3,6 ~I
2	(8)	Gb	7 DN
9	(9)	Gb	Assumption
2	(10)	~Gb	9,6 ~I
2	(11)	Gb&~Gb	8,10 &I
2	(12)	(∀x)(Gx&~Gx)	11 ∀I
1	(13)	(∀x)(Gx&~Gx)	1,2,12 ∃E

Proof of (7)

Problem is: (∀x)(Fx→(∀y)~Fy) ⊢ ~(∃x)Fx

1	(1)	(∀x)(Fx→(∀y)~Fy)	Premise
2	(2)	(∃x)Fx	Assumption
3	(3)	Fa	Assumption
1	(4)	Fa→(∀y)~Fy	1 ∀E
1,3	(5)	(∀y)~Fy	4,3 →E
1,3	(6)	~Fa	5 ∀E
1,3	(7)	Λ	6,3 ~E
1,2	(8)	Λ	2,3,7 ∃E
1	(9)	~(∃x)Fx	2,8 ~I

Proof of (8)

Problem is: (∀x)(∃y)(Fx&Gy) ⊢ (∃y)(∀x)(Fx&Gy)

1	(1)	(∀x)(∃y)(Fx&Gy)	Premise
1	(2)	(∃y)(Fa&Gy)	1 ∀E
3	(3)	Fa&Gb	Assumption
1	(4)	(∃y)(Fc&Gy)	1 ∀E
5	(5)	Fc&Gd	Assumption
5	(6)	Fc	5 &E
1	(7)	Fc	4,5,6 ∃E
3	(8)	Gb	3 &E
1,3	(9)	Fc&Gb	7,8 &I
1,3	(10)	(∀x)(Fx&Gb)	9 ∀I
1,3	(11)	(∃y)(∀x)(Fx&Gy)	10 ∃I
1	(12)	(∃y)(∀x)(Fx&Gy)	2,3,11 ∃E

Proof of (9)

Problem is: (∃y)(∀x)(Fx&Gy) ⊢ (∀x)(∃y)(Fx&Gy)

1	(1)	(∃y)(∀x)(Fx&Gy)	Premise
2	(2)	(∀x)(Fx&Gb)	Assumption
2	(3)	Fa&Gb	2 ∀E
2	(4)	(∃y)(Fa&Gy)	3 ∃I
1	(5)	(∃y)(Fa&Gy)	1,2,4 ∃E
1	(6)	(∀x)(∃y)(Fx&Gy)	5 ∀I