Announcements and Such

- Today's Music: *The Rolling Stones*
- I have posted my solutions to HW #4 (with the shortest proofs I know).
- HW #5 is due today @ 4pm.
  - I've posted a handout entitled "Working with LMPL Interpretations", which contains model answers for LMPL semantics problems.
- HW #6 has been posted, and will be due next Thursday @ 4pm.

☞ **The final is *in class* next Thursday. You'll be given 3 hours to do it.**

- I've posted two important handouts concerning the final exam:
  - The (Complete) Natural Deduction Rules Handout (provided at final).
  - A sample final exam, which has the same structure as the actual final. This sample will be discussed, in detail, in lecture tomorrow.
- Today: Chapter 6 — Natural deduction proofs in LMPL

The Rule of ∃-Introduction

Rule of ∃-Introduction: For any sentence $\phi\tau$, if $\phi\tau$ has been inferred at line j in a proof, then at line k we may infer $\ulcorner(\exists\nu)\phi\nu\urcorner$, labeling the line 'j ∃I' and writing on its left the numbers that occur on the left of j.

$$\begin{array}{llll} a_1,\ldots,a_n & (j) & \phi\tau & \\ & \vdots & & \\ a_1,\ldots,a_n & (k) & (\exists\nu)\phi\nu & j\ \exists I \end{array}$$

Where $\ulcorner(\exists\nu)\phi\nu\urcorner$ is obtained syntactically from $\phi\tau$ by:

- Replacing ***one or more occurrences*** of τ in $\phi\tau$ by a *single* variable ν.
- Note: the variable ν ***must not already occur in*** the expression $\phi\tau$. [This prevents *double-binding*, *e.g.*, '$(\exists x)(\exists x)(Fx \,\&\, Gx)$'.]
- And, finally, prefixing the quantifier $\ulcorner(\exists\nu)\urcorner$ in front of the resulting expression (which may now have both $\ulcorner\nu\urcorner$s and $\ulcorner\tau\urcorner$s occurring in it).

The Rule of ∀-Elimination

Rule of ∀-Elimination: For any sentence $\ulcorner(\forall\nu)\phi\nu\urcorner$ and constant τ, if $\ulcorner(\forall\nu)\phi\nu\urcorner$ has been inferred at a line j, then at line k we may infer $\phi\tau$, labeling the line 'j ∀E' and writing on its left the numbers that appear on the left of j.

$$\begin{array}{llll} a_1,\ldots,a_n & (j) & (\forall\nu)\phi\nu & \\ & \vdots & & \\ a_1,\ldots,a_n & (k) & \phi\tau & j\ \forall E \end{array}$$

Where $\phi\tau$ is obtained syntactically from $\ulcorner(\forall\nu)\phi\nu\urcorner$ by:

- Deleting the quantifier prefix $\ulcorner(\forall\nu)\urcorner$.
- Replacing ***every occurrence*** of ν in the open sentence $\phi\nu$ by ***one and the same*** constant τ. [This prevents *fallacies*, *e.g.*, $(\forall x)(Fx \to Gx) \nvDash Fa \to Gb$.]
- Note: since '∀' means *everything*, there are *no* restrictions on *which* individual constant may be used in an application of ∀E.

The Rule of ∀-Introduction: Some Background

- It is useful to think of a universal claim $\ulcorner(\forall\nu)\phi\nu\urcorner$ as a *conjunction* which asserts that the predicate expression ϕ is satisfied by *all objects* in the domain of discourse (*i.e.*, the conjunction $\ulcorner\phi a \,\&\, (\phi b \,\&\, (\phi c \,\&\, \ldots))\urcorner$ is true).
- So, in order to be able to *introduce* the universal quantifier (*i.e.*, to *legitimately infer* $\ulcorner(\forall\nu)\phi\nu\urcorner$ in a proof), we must be in a position to prove $\phi\tau$, for *any* individual constant τ. This is called *generalizable reasoning*.
- Consider the following *legitimate* introduction of a universal claim:

Problem is: (∀x)(Fx→Gx), (∀x)Fx ⊢ (∀x)Gx

1	(1)	(∀x)(Fx→Gx)	Premise
2	(2)	(∀x)Fx	Premise
1	(3)	Fa→Ga	1 ∀E
2	(4)	Fa	2 ∀E
1,2	(5)	Ga	3,4 →E
1,2	(6)	(∀x)Gx	5 ∀I

The Rule of ∀-Introduction: II

- We can legitimately infer '$(\forall x)Gx$' at line 6 of this proof, because our inference to 'Gb' is *generalizable* — *i.e.*, we could have deduced $\ulcorner G\tau \urcorner$, for *any* individual constant τ — using *exactly parallel* reasoning.
- However, consider the following *il*legitimate "∀-Introduction" step:

1	(1)	(∀x)(Fx→Gx)	Premise	
2	(2)	Fb	Premise	
1	(3)	Fb→Gb	1 ∀E	
1,2	(4)	Gb	2,3 →E	
1,2	(5)	(∀x)Gx	4 ∀I	NO!!

- This is *not* a valid inference, since $(\forall x)(Fx \to Gx), Fb \nvDash (\forall x)Gx$!
- So, what went wrong? The problem is that the inference to 'Gb' at (4) is *not* generalizable. We can *not* deduce $\ulcorner G\tau \urcorner$ — for *any* τ — from the premises '$(\forall x)(Fx \to Gx)$' and 'Fb'. We can *only* infer '$G\boldsymbol{b}$'.

The Rule of ∀-Introduction: III

Rule of ∀-Introduction: For any sentence $\phi\tau$, if $\phi\tau$ has been inferred at a line j, then *provided that* τ *does not occur in any premise or assumption whose line number is on the left at line* j, we may infer $\ulcorner(\forall\nu)\phi\nu\urcorner$ at line k, labeling the line 'j ∀I' and writing on its left the same numbers as occur on the left at line j.

$$\begin{array}{lll} a_1,\ldots,a_n & (j) & \phi\tau \\ & \vdots & \\ a_1,\ldots,a_n & (k) & (\forall\nu)\phi\nu \quad j\ \forall I \end{array}$$

Where $\ulcorner(\forall\nu)\phi\nu\urcorner$ is obtained by:

- Replacing ***every*** occurrence of τ in $\phi\tau$ with ν and prefixing $\ulcorner(\forall\nu)\urcorner$. [Again, 'every' prevents *fallacies*, *e.g.*, $(\forall x)(Fx \to Gx) \nvDash (\forall x)(\forall y)(Fx \to Gy)$.]
- τ ***does not occur in*** any of the formulae $a_1,\ldots,a_n$. [ensures *generalizability*]
- ν ***does not occur in*** $\phi\tau$. [prevents *double-binding*]

The Rule of ∀-Introduction: Four Examples

- Here are four examples of LMPL sequents involving the three quantifier rules we've learned so far (∃I, ∀E, and ∀I).

(1) $(\forall x)(Fx \to Gx) \vdash (\forall x)Fx \to (\forall x)Gx$

(2) $\sim(\exists x)(Fx \,\&\, Gx) \vdash (\forall x)(Fx \to \sim Gx)$

(3) $\sim(\forall x)Fx \vdash (\exists x)\sim Fx$

(4) $(\forall x)[Fx \to (\forall y)Gy] \vdash (\forall x)(\forall y)(Fx \to Gy)$

Proof of (1)

Problem is: (∀x)(Fx→Gx) ⊢ (∀x)Fx→(∀x)Gx

1	(1)	(∀x)(Fx→Gx)	Premise
2	(2)	(∀x)Fx	Assumption
1	(3)	Fa→Ga	1 ∀E
2	(4)	Fa	2 ∀E
1,2	(5)	Ga	3,4 →E
1,2	(6)	(∀x)Gx	5 ∀I
1	(7)	(∀x)Fx→(∀x)Gx	2,6 →I

Proof of (2)

Problem is: ~(∃x)(Fx&Gx) ⊢ (∀x)(Fx→~Gx)

1	(1)	~(∃x)(Fx&Gx)	Premise
2	(2)	Fa	Assumption
3	(3)	Ga	Assumption
2,3	(4)	Fa&Ga	2,3 &I
2,3	(5)	(∃x)(Fx&Gx)	4 ∃I
1,2,3	(6)	⋏	1,5 ~E
1,2	(7)	~Ga	3,6 ~I
1	(8)	Fa→~Ga	2,7 →I
1	(9)	(∀x)(Fx→~Gx)	8 ∀I

Proof of (3)

Problem is: ~(∀x)Fx ⊢ (∃x)~Fx

1	(1)	~(∀x)Fx	Premise
2	(2)	~(∃x)~Fx	Assumption
3	(3)	~Fa	Assumption
3	(4)	(∃x)~Fx	3 ∃I
2,3	(5)	⋏	2,4 ~E
2	(6)	~~Fa	3,5 ~I
2	(7)	Fa	6 DN
2	(8)	(∀x)Fx	7 ∀I
1,2	(9)	⋏	1,8 ~E
1	(10)	~~(∃x)~Fx	2,9 ~I
1	(11)	(∃x)~Fx	10 DN

Proof of (4)

Problem is: (∀x)(Fx→(∀y)Gy) ⊢ (∀x)(∀y)(Fx→Gy)

1	(1)	(∀x)(Fx→(∀y)Gy)	Premise
2	(2)	Fa	Assumption
1	(3)	Fa→(∀y)Gy	1 ∀E
1,2	(4)	(∀y)Gy	3,2 →E
1,2	(5)	Gb	4 ∀E
1	(6)	Fa→Gb	2,5 →I
1	(7)	(∀y)(Fa→Gy)	6 ∀I
1	(8)	(∀x)(∀y)(Fx→Gy)	7 ∀I

The Rule of ∃-Elimination: Some Background

- It is useful to think of an existential claim $\ulcorner(\exists \nu)\phi\nu\urcorner$ as a *disjunction* which asserts that the predicate expression ϕ is satisfied by *at least one* object in the domain (*i.e.*, that the disjunction $\ulcorner\phi a \vee (\phi b \vee (\phi c \vee \ldots))\urcorner$ is true).
- In this way, we would expect the elimination rule for ∃ to be similar to the elimination rule for ∨. That is, we'd expect the ∃E rule to be similar to the ∨E rule. Indeed, this is the case. It's best to start with a simple example.
- Consider the following *legitimate* elimination of an existential claim:

Problem is: (∃x)(Fx&Gx) ⊢ (∃x)Fx

1	(1)	(∃x)(Fx&Gx)	Premise
2	(2)	Fa&Ga	Assumption
2	(3)	Fa	2 &E
2	(4)	(∃x)Fx	3 ∃I
1	(5)	(∃x)Fx	1,2,4 ∃E

The Rule of ∃-Elimination: II

- To derive a sentence $\mathscr{P}$ using the ∃E rule (with some existential sentence $\ulcorner(\exists\nu)\phi\nu\urcorner$), we must first *assume* an *instance* $\phi\tau$ of $\ulcorner(\exists\nu)\phi\nu\urcorner$.
- If we can deduce $\mathscr{P}$ from this assumed instance $\phi\tau$ — *using* ***generalizable*** *reasoning* — then we may infer $\mathscr{P}$ *outright*.
- It is because our reasoning from the *instance* $\phi\tau$ of $\ulcorner(\exists\nu)\phi\nu\urcorner$ to $\mathscr{P}$ *does not depend on our choice of constant* τ (*i.e.*, that our reasoning from $\phi\tau$ to $\mathscr{P}$ is *generalizable*) that makes this inference valid.
- When our reasoning is generalizable in this sense, it's as if we are showing that $\mathscr{P}$ can be deduced from *any* instance $\phi\tau$ of $\ulcorner(\exists\nu)\phi\nu\urcorner$.
- As such, this is just like showing that $\mathscr{P}$ can be deduced from *any disjunct* of the disjunction $\ulcorner\phi a \vee (\phi b \vee (\phi c \vee \ldots))\urcorner$. And, this is just like ∨E reasoning (except that ∃E only requires *one* assumption).

The Rule of ∃-Elimination: III

- Here's an *il*legitimate "∃-Elimination" step:

1	(1)	(∃x)Fx	Premise	
2	(2)	Ga	Premise	
3	(3)	Fa	Assumption	
2,3	(4)	Fa&Ga	2,3 &I	
2,3	(5)	(∃x)(Fx&Gx)	4 ∃I	
1,2	(6)	(∃x)(Fx&Gx)	1,3,5 ∃E	NO!!

- This is *not* a valid inference: $(\exists x)Fx, Ga \nvDash (\exists x)(Fx \,\&\, Gx)$!
- So, what went wrong here? The problem is that the inference to '$(\exists x)(Fx \,\&\, Gx)$' at line (5) does *not* use *generalizable* reasoning.
- We can *not* legitimately infer '$(\exists x)(Fx \,\&\, Gx)$' at line (5) from an *arbitrary instance* $\ulcorner F\tau\urcorner$ of '$(\exists x)Fx$'. We *must* assume '$F\boldsymbol{a}$' *in particular* at line (3) in order to deduce '$(\exists x)(Fx \,\&\, Gx)$' at line (5).

The Rule of ∃-Elimination: Official Definition

∃-Elimination: If $\ulcorner(\exists\nu)\phi\nu\urcorner$ occurs at i depending on $a_1,\ldots,a_n$, an instance $\phi\tau$ of $\ulcorner(\exists\nu)\phi\nu\urcorner$ is *assumed* at j, and $\mathscr{P}$ is inferred at k depending on $b_1,\ldots,b_u$, then at line m we may infer $\mathscr{P}$, with label 'i, j, k ∃E' and dependencies $\{a_1,\ldots,a_n\} \cup \{b_1,\ldots,b_u\}/j$:

$a_1,\ldots,a_n$	(i)	$(\exists\nu)\phi\nu$	
	⋮		
j	(j)	$\phi\tau$	Assumption
	⋮		
$b_1,\ldots,b_u$	(k)	$\mathscr{P}$	
	⋮		
$\{a_1,\ldots,a_n\} \cup \{b_1,\ldots,b_u\}/j$	(m)	$\mathscr{P}$	i, j, k ∃E

Provided that ***all four*** of the following conditions are met:

- τ (in $\phi\tau$) replaces ***every*** occurrence of ν in $\phi\nu$. [avoids fallacies]
- τ ***does not occur in*** $\ulcorner(\exists\nu)\phi\nu\urcorner$. [generalizability]
- τ ***does not occur in*** $\mathscr{P}$. [generalizability]
- τ ***does not occur in*** *any* of $b_1,\ldots,b_u$, except (possibly) $\phi\tau$ itself. [generalizability]

The Rule of ∃-Elimination: Nine Examples

- Here are 9 examples of proofs involving all four quantifier rules.

1. $(\exists x){\sim}Fx \vdash {\sim}(\forall x)Fx$ [*p*. 200, example 5]
2. $(\exists x)(Fx \to A) \vdash (\forall x)Fx \to A$ [*p*. 201, example 6]
3. $(\forall x)(\forall y)(Gy \to Fx) \vdash (\forall x)[(\exists y)Gy \to Fx]$ [*p*. 203, I. # 19 ⇒]
4. $(\exists x)[Fx \to (\forall y)Gy] \vdash (\exists x)(\forall y)(Fx \to Gy)$ [*p*. 203, I. # 20 ⇐]
5. $A \vee (\exists x)Fx \vdash (\exists x)(A \vee Fx)$ [*p*. 203, II. # 2 ⇐]
6. $(\exists x)(Fx \,\&\, {\sim}Fx) \vdash (\forall x)(Gx \,\&\, {\sim}Gx)$ [*p*. 203, I. # 12 ⇒]
7. $(\forall x)[Fx \to (\forall y){\sim}Fy] \vdash {\sim}(\exists x)Fx$ [*p*. 203, I. # 5]
8. $(\forall x)(\exists y)(Fx \,\&\, Gy) \vdash (\exists y)(\forall x)(Fx \,\&\, Gy)$ [*p*. 201, example 7]
9. $(\exists y)(\forall x)(Fx \,\&\, Gy) \vdash (\forall x)(\exists y)(Fx \,\&\, Gy)$ [other direction]

Proof of (1)

Problem is: (∃x)~Fx ⊦ ~(∀x)Fx

1	(1)	(∃x)~Fx	Premise
2	(2)	(∀x)Fx	Assumption
3	(3)	~Fa	Assumption
2	(4)	Fa	2 ∀E
2,3	(5)	Λ	3,4 ~E
1,2	(6)	Λ	1,3,5 ∃E
1	(7)	~(∀x)Fx	2,6 ~I

Proof of (2)

Problem is: (∃x)(Fx→A) ⊦ (∀x)Fx→A

1	(1)	(∃x)(Fx→A)	Premise
2	(2)	(∀x)Fx	Assumption
3	(3)	Fa→A	Assumption
2	(4)	Fa	2 ∀E
2,3	(5)	A	3,4 →E
1,2	(6)	A	1,3,5 ∃E
1	(7)	(∀x)Fx→A	2,6 →I

Proof of (3)

Problem is: (∀x)(∀y)(Gy→Fx) ⊦ (∀x)((∃y)Gy→Fx)

1	(1)	(∀x)(∀y)(Gy→Fx)	Premise
2	(2)	(∃y)Gy	Assumption
3	(3)	Gb	Assumption
1	(4)	(∀y)(Gy→Fa)	1 ∀E
1	(5)	Gb→Fa	4 ∀E
1,3	(6)	Fa	5,3 →E
1,2	(7)	Fa	2,3,6 ∃E
1	(8)	(∃y)Gy→Fa	2,7 →I
1	(9)	(∀x)((∃y)Gy→Fx)	8 ∀I

Proof of (4)

Problem is: (∃x)(Fx→(∀y)Gy) ⊦ (∃x)(∀y)(Fx→Gy)

1	(1)	(∃x)(Fx→(∀y)Gy)	Premise
2	(2)	Fa→(∀y)Gy	Assumption
3	(3)	Fa	Assumption
2,3	(4)	(∀y)Gy	2,3 →E
2,3	(5)	Gb	4 ∀E
2	(6)	Fa→Gb	3,5 →I
2	(7)	(∀y)(Fa→Gy)	6 ∀I
2	(8)	(∃x)(∀y)(Fx→Gy)	7 ∃I
1	(9)	(∃x)(∀y)(Fx→Gy)	1,2,8 ∃E

Proof of (5)

Problem is: A∨(∃x)Fx ⊦ (∃x)(A∨Fx)

1	(1)	A∨(∃x)Fx	Premise
2	(2)	A	Assumption
2	(3)	A∨Fa	2 ∨I
2	(4)	(∃x)(A∨Fx)	3 ∃I
5	(5)	(∃x)Fx	Assumption
6	(6)	Fa	Assumption
6	(7)	A∨Fa	6 ∨I
6	(8)	(∃x)(A∨Fx)	7 ∃I
5	(9)	(∃x)(A∨Fx)	5,6,8 ∃E
1	(10)	(∃x)(A∨Fx)	1,2,4,5,9 ∨E

Proof of (6)

Problem is: (∃x)(Fx&~Fx) ⊦ (∀x)(Gx&~Gx)

1	(1)	(∃x)(Fx&~Fx)	Premise
2	(2)	Fa&~Fa	Assumption
3	(3)	~Gb	Assumption
2	(4)	~Fa	2 &E
2	(5)	Fa	2 &E
2	(6)	Λ	4,5 ~E
2	(7)	~~Gb	3,6 ~I
2	(8)	Gb	7 DN
9	(9)	Gb	Assumption
2	(10)	~Gb	9,6 ~I
2	(11)	Gb&~Gb	8,10 &I
2	(12)	(∀x)(Gx&~Gx)	11 ∀I
1	(13)	(∀x)(Gx&~Gx)	1,2,12 ∃E

Proof of (7)

Problem is: (∀x)(Fx→(∀y)~Fy) ⊦ ~(∃x)Fx

1	(1)	(∀x)(Fx→(∀y)~Fy)	Premise
2	(2)	(∃x)Fx	Assumption
3	(3)	Fa	Assumption
1	(4)	Fa→(∀y)~Fy	1 ∀E
1,3	(5)	(∀y)~Fy	4,3 →E
1,3	(6)	~Fa	5 ∀E
1,3	(7)	Λ	6,3 ~E
1,2	(8)	Λ	2,3,7 ∃E
1	(9)	~(∃x)Fx	2,8 ~I

Proof of (8)

Problem is: (∀x)(∃y)(Fx&Gy) ⊦ (∃y)(∀x)(Fx&Gy)

1	(1)	(∀x)(∃y)(Fx&Gy)	Premise
1	(2)	(∃y)(Fa&Gy)	1 ∀E
3	(3)	Fa&Gb	Assumption
1	(4)	(∃y)(Fc&Gy)	1 ∀E
5	(5)	Fc&Gd	Assumption
5	(6)	Fc	5 &E
1	(7)	Fc	4,5,6 ∃E
3	(8)	Gb	3 &E
1,3	(9)	Fc&Gb	7,8 &I
1,3	(10)	(∀x)(Fx&Gb)	9 ∀I
1,3	(11)	(∃y)(∀x)(Fx&Gy)	10 ∃I
1	(12)	(∃y)(∀x)(Fx&Gy)	2,3,11 ∃E

Proof of (9)

Problem is: (∃y)(∀x)(Fx&Gy) ⊦ (∀x)(∃y)(Fx&Gy)

1	(1)	(∃y)(∀x)(Fx&Gy)	Premise
2	(2)	(∀x)(Fx&Gb)	Assumption
2	(3)	Fa&Gb	2 ∀E
2	(4)	(∃y)(Fa&Gy)	3 ∃I
1	(5)	(∃y)(Fa&Gy)	1,2,4 ∃E
1	(6)	(∀x)(∃y)(Fx&Gy)	5 ∀I

Two LMPL Extensions of Sequent Introduction

- Here are two additions to our list of SI sequents:

(QS) One can infer ⌜$(\forall x)\sim\phi x$⌝ from (the *logically equivalent* sentence) ⌜$\sim(\exists x)\phi x$⌝, and *vice versa*; and, that one can infer ⌜$(\exists x)\sim\phi x$⌝ from (the *logically equivalent*) ⌜$\sim(\forall x)\phi x$⌝, and *vice versa*.

$$(\forall x)\sim\phi x \dashv\vdash \sim(\exists x)\phi x\text{; and, } (\exists x)\sim\phi x \dashv\vdash \sim(\forall x)\phi x \qquad \text{(QS)}$$

(AV) One can infer a *closed* LMPL sentence ψ from (the *logically equivalent* sentence) ψ', and *vice versa*, where ψ and ψ' are *alphabetic variants*. Two formulas are *alphabetic variants* if and only if they differ *only* in a (conventional) choice of individual *variable* letters (*not* kosher for constants!). *E.g.*, '$(\forall x)Fx$' and '$(\forall y)Fy$' are (closed) *alphabetic variants*, because they differ *only* in which individual variable ('x' or 'y') is used, but they have the same *logical* (*i.e.*, *syntactical*) *structure*.

$$\psi \dashv\vdash \psi' \qquad \text{(AV)}$$

Our (New) Official List of Sequents and Theorems (see pp. 123, 204, and 206)

(DS)	$A \vee B, \sim A \vdash B$; or; $A \vee B, \sim B \vdash A$	(Imp)	$A \to B \dashv\vdash \sim A \vee B$
(MT)	$A \to B, \sim B \vdash \sim A$	(Neg-Imp)	$\sim(A \to B) \dashv\vdash A \,\&\, \sim B$
(PMI)	$A \vdash B \to A$	(Dist)	$A \,\&\, (B \vee C) \dashv\vdash (A \,\&\, B) \vee (A \,\&\, C)$
(PMI)	$\sim A \vdash A \to B$	(Dist)	$A \vee (B \,\&\, C) \dashv\vdash (A \vee B) \,\&\, (A \vee C)$
(DN⁺)	$A \vdash \sim\sim A$	(EFQ, or ⋏E)	$\curlywedge \vdash A$
(DEM)	$\sim(A \,\&\, B) \dashv\vdash \sim A \vee \sim B$	(Com)	$A * B \vdash B * A$
(DEM)	$\sim(A \vee B) \dashv\vdash \sim A \,\&\, \sim B$	(SDN)	$\sim\sim A * \sim\sim B \dashv\vdash A * B$
(DEM)	$\sim(\sim A \vee \sim B) \dashv\vdash A \,\&\, B$	(SDN)	$A * B \dashv\vdash \sim\sim A * B \dashv\vdash A * \sim\sim B$
(DEM)	$\sim(\sim A \,\&\, \sim B) \dashv\vdash A \vee B$	(LEM)	$\vdash A \vee \sim A$
(QS)	$(\forall x)\sim\phi x \dashv\vdash \sim(\exists x)\phi x$	(QS)	$(\exists x)\sim\phi x \dashv\vdash \sim(\forall x)\phi x$
		(AV)	$\psi \dashv\vdash \psi'$

In (Com), '$*$' can be any binary connective *except* '→'. In (SDN), '$*$' can be *any* binary connective. In (AV), ψ must be *closed*, and ψ' must be an *alphabetic variant* of ψ.

The Value of (QS) — Its Four Simplest Instances

(∀x)~Fx ⊦ ~(∃x)Fx

1	(1)	(∀x)~Fx	Premise
2	(2)	(∃x)Fx	Ass
3	(3)	Fa	Ass
1	(4)	~Fa	1 ∀E
1,3	(5)	⋏	4,3 ~E
1,2	(6)	⋏	2,3,5 ∃E
1	(7)	~(∃x)Fx	2,6 ~I

~(∃x)Fx ⊦ (∀x)~Fx

1	(1)	~(∃x)Fx	Premise
2	(2)	Fa	Ass
2	(3)	(∃x)Fx	2 ∃I
1,2	(4)	⋏	1,3 ~E
1	(5)	~Fa	2,4 ~I
1	(6)	(∀x)~Fx	5 ∀I

(∃x)~Fx ⊦ ~(∀x)Fx

1	(1)	(∃x)~Fx	Premise
2	(2)	(∀x)Fx	Ass
3	(3)	~Fa	Ass
2	(4)	Fa	2 ∀E
2,3	(5)	⋏	3,4 ~E
1,2	(6)	⋏	1,3,5 ∃E
1	(7)	~(∀x)Fx	2,6 ~I

~(∀x)Fx ⊦ (∃x)~Fx

1	(1)	~(∀x)Fx	Premise
2	(2)	~(∃x)~Fx	Ass
3	(3)	~Fa	Ass
3	(4)	(∃x)~Fx	3 ∃I
2,3	(5)	⋏	2,4 ~E
2	(6)	~~Fa	3,5 ~I
2	(7)	Fa	6 DN
2	(8)	(∀x)Fx	7 ∀I
1,2	(9)	⋏	1,8 ~E
1	(10)	~~(∃x)~Fx	2,9 ~I
1	(11)	(∃x)~Fx	10 DN

Three Examples Involving the LMPL SI Extension (QS)

- Here are three examples of proofs involving SI (QS):
  1. $\sim(\forall x)\sim Fx \vdash (\exists x)Fx$ [p. 207, #7 ⇐]
  2. $\sim(\exists x)(Fx \,\&\, Gx) \vee (\exists x)\sim Gx, (\forall y)Gy \vdash (\forall z)(Fz \rightarrow \sim Gz)$ [p. 205, ex. 1]
  3. $(\forall x)Fx \rightarrow A \vdash (\exists x)(Fx \rightarrow A)$ [p. 205, ex. 2]

Proof of (1)

1	(1)	$\sim(\forall x)\sim Fx$	Premise
2	(2)	$\sim(\exists x)Fx$	Assumption
2	(3)	$(\forall x)\sim Fx$	2 SI (QS)
1,2	(4)	$\curlywedge$	1, 3 ~E
1	(5)	$\sim\sim(\exists x)Fx$	2, 4 ~I
1	(6)	$(\exists x)Fx$	5 DN

Proof of (2)

1	(1)	$\sim(\exists x)(Fx \,\&\, Gx) \vee (\exists x)\sim Gx$	Premise
2	(2)	$(\forall y)Gy$	Premise
3	(3)	$\sim(\exists x)(Fx \,\&\, Gx)$	Assumption
3	(4)	$(\forall x)\sim(Fx \,\&\, Gx)$	3 SI (QS)
3	(5)	$\sim(Fa \,\&\, Ga)$	4 ∀E
3	(6)	$\sim Fa \vee \sim Ga$	5 SI (DeM)
3	(7)	$Fa \rightarrow \sim Ga$	6 SI (Imp)
3	(8)	$(\forall z)(Fz \rightarrow \sim Gz)$	7 ∀I
9	(9)	$(\exists x)\sim Gx$	Assumption
10	(10)	$\sim Ga$	Assumption
2	(11)	Ga	2 ∀E
2,10	(12)	$\curlywedge$	10, 11 ~E
2,10	(13)	$(\forall z)(Fz \rightarrow \sim Gz)$	12 SI (EFQ)
2,9	(14)	$(\forall z)(Fz \rightarrow \sim Gz)$	9, 10, 13 ∃E
1,2	(15)	$(\forall z)(Fz \rightarrow \sim Gz)$	1, 3, 8, 9, 14 ∨E

Proof of (3)

Problem is: $(\forall x)Fx \rightarrow A \vdash (\exists x)(Fx \rightarrow A)$

1	(1)	$(\forall x)Fx \rightarrow A$	Premise
1	(2)	$\sim(\forall x)Fx \vee A$	1 SI (Imp)
3	(3)	$\sim(\forall x)Fx$	Assumption
3	(4)	$(\exists x)\sim Fx$	3 SI (QS)
5	(5)	$\sim Fa$	Assumption
5	(6)	$Fa \rightarrow A$	5 SI (PMI)
5	(7)	$(\exists x)(Fx \rightarrow A)$	6 ∃I
3	(8)	$(\exists x)(Fx \rightarrow A)$	4,5,7 ∃E
9	(9)	A	Assumption
9	(10)	$Fa \rightarrow A$	9 SI (PMI)
9	(11)	$(\exists x)(Fx \rightarrow A)$	10 ∃I
1	(12)	$(\exists x)(Fx \rightarrow A)$	2,3,8,9,11 ∨E

The Value of (AV)

- Here are the two simplest instances of (AV):

$(\forall x)Fx \vdash (\forall y)Fy$				$(\exists x)Fx \vdash (\exists y)Fy$			
1	(1)	$(\forall x)Fx$	Premise	1	(1)	$(\exists x)Fx$	Premise
1	(2)	Fa	1 $\forall$E	2	(2)	Fa	Ass
1	(3)	$(\forall y)Fy$	2 $\forall$I	2	(3)	$(\exists y)Fy$	2 $\exists$I
				1	(4)	$(\exists y)Fy$	1,2,3 $\exists$E

- Here's an (AV)-aided proof of the following sequent

$$(\forall x)Fx, (\forall y)Fy \to (\forall y)Gy \vdash (\forall z)Gz$$

1	(1)	$(\forall x)Fx$	Premise
2	(2)	$(\forall y)Fy \to (\forall y)Gy$	Premise
1	(3)	$(\forall y)Fy$	1 SI (AV)
1,2	(4)	$(\forall y)Gy$	2,3 $\to$E
1,2	(5)	$(\forall z)Gz$	4 SI (AV)