Announcements and Such

- Today's Music: *Wishbone Ash*
- I have posted my solutions to HW #4 and HW #5.
- HW #6 is due on Thursday @ 4pm.
- ☞ **The final is *in class* on Thursday. You'll be given 3 hours to do it.**
- I've posted two important handouts concerning the final exam:
  - The (Complete) Natural Deduction Rules Handout (provided at final).
  - A sample final exam, which has the same structure as the actual final. This sample was discussed, in detail, in lecture last week.
- Today: Chapter 6, finalé, and Chapters 7 & 8 Intro.
  - We'll finish-up chapter 6 (LMPL) today, and move on to Chs. 7&8.
  - I will only be covering (some of) the **L2PL** parts of Chapters 7 & 8.

The Rule of ∃-Elimination: Nine Examples

- Here are 9 examples of proofs involving all four quantifier rules.

1. $(\exists x){\sim}Fx \vdash {\sim}(\forall x)Fx$ [*p*. 200, example 5]
2. $(\exists x)(Fx \rightarrow A) \vdash (\forall x)Fx \rightarrow A$ [*p*. 201, example 6]
3. $(\forall x)(\forall y)(Gy \rightarrow Fx) \vdash (\forall x)[(\exists y)Gy \rightarrow Fx]$ [*p*. 203, I. # 19 ⇒]
4. $(\exists x)[Fx \rightarrow (\forall y)Gy] \vdash (\exists x)(\forall y)(Fx \rightarrow Gy)$ [*p*. 203, I. # 20 ⇐]
5. $A \vee (\exists x)Fx \vdash (\exists x)(A \vee Fx)$ [*p*. 203, II. # 2 ⇐]
6. $(\exists x)(Fx \,\&\, {\sim}Fx) \vdash (\forall x)(Gx \,\&\, {\sim}Gx)$ [*p*. 203, I. # 12 ⇒]
7. $(\forall x)[Fx \rightarrow (\forall y){\sim}Fy] \vdash {\sim}(\exists x)Fx$ [*p*. 203, I. # 5]
8. $(\forall x)(\exists y)(Fx \,\&\, Gy) \vdash (\exists y)(\forall x)(Fx \,\&\, Gy)$ [*p*. 201, example 7]
9. $(\exists y)(\forall x)(Fx \,\&\, Gy) \vdash (\forall x)(\exists y)(Fx \,\&\, Gy)$ [other direction]

Proof of (8)

Problem is: (∀x)(∃y)(Fx&Gy) ⊢ (∃y)(∀x)(Fx&Gy)

1	(1)	(∀x)(∃y)(Fx&Gy)	Premise
1	(2)	(∃y)(Fa&Gy)	1 ∀E
3	(3)	Fa&Gb	Assumption
1	(4)	(∃y)(Fc&Gy)	1 ∀E
5	(5)	Fc&Gd	Assumption
5	(6)	Fc	5 &E
1	(7)	Fc	4,5,6 ∃E
3	(8)	Gb	3 &E
1,3	(9)	Fc&Gb	7,8 &I
1,3	(10)	(∀x)(Fx&Gb)	9 ∀I
1,3	(11)	(∃y)(∀x)(Fx&Gy)	10 ∃I
1	(12)	(∃y)(∀x)(Fx&Gy)	2,3,11 ∃E

Proof of (9)

Problem is: (∃y)(∀x)(Fx&Gy) ⊢ (∀x)(∃y)(Fx&Gy)

1	(1)	(∃y)(∀x)(Fx&Gy)	Premise
2	(2)	(∀x)(Fx&Gb)	Assumption
2	(3)	Fa&Gb	2 ∀E
2	(4)	(∃y)(Fa&Gy)	3 ∃I
1	(5)	(∃y)(Fa&Gy)	1,2,4 ∃E
1	(6)	(∀x)(∃y)(Fx&Gy)	5 ∀I

Two LMPL Extensions of Sequent Introduction

- Here are two additions to our list of SI sequents:

(QS) One can infer ⌜$(\forall x){\sim}\phi x$⌝ from (the *logically equivalent* sentence) ⌜${\sim}(\exists x)\phi x$⌝, and *vice versa*; and, that one can infer ⌜$(\exists x){\sim}\phi x$⌝ from (the *logically equivalent*) ⌜${\sim}(\forall x)\phi x$⌝, and *vice versa*.

$$(\forall x){\sim}\phi x \dashv\vdash {\sim}(\exists x)\phi x\text{; and, } (\exists x){\sim}\phi x \dashv\vdash {\sim}(\forall x)\phi x \qquad \text{(QS)}$$

(AV) One can infer a *closed* LMPL sentence ψ from (the *logically equivalent* sentence) ψ', and *vice versa*, where ψ and ψ' are *alphabetic variants*. Two formulas are *alphabetic variants* if and only if they differ *only* in a (conventional) choice of individual *variable* letters (*not* kosher for constants!). *E.g.*, '$(\forall x)Fx$' and '$(\forall y)Fy$' are (closed) *alphabetic variants*, because they differ *only* in which individual variable ('x' or 'y') is used, but they have the same *logical* (*i.e.*, *syntactical*) *structure*.

$$\psi \dashv\vdash \psi' \qquad \text{(AV)}$$

Our (New) Official List of Sequents and Theorems (see pp. 123, 204, and 206)

(DS)	$A \vee B, {\sim}A \vdash B$; or; $A \vee B, {\sim}B \vdash A$	(Imp)	$A \to B \dashv\vdash {\sim}A \vee B$
(MT)	$A \to B, {\sim}B \vdash {\sim}A$	(Neg-Imp)	${\sim}(A \to B) \dashv\vdash A \,\&\, {\sim}B$
(PMI)	$A \vdash B \to A$	(Dist)	$A \,\&\, (B \vee C) \dashv\vdash (A \,\&\, B) \vee (A \,\&\, C)$
(PMI)	${\sim}A \vdash A \to B$	(Dist)	$A \vee (B \,\&\, C) \dashv\vdash (A \vee B) \,\&\, (A \vee C)$
(DN$^+$)	$A \vdash {\sim}{\sim}A$	(EFQ, or $\curlywedge$E)	$\curlywedge \vdash A$
(DEM)	${\sim}(A \,\&\, B) \dashv\vdash {\sim}A \vee {\sim}B$	(Com)	$A * B \vdash B * A$
(DEM)	${\sim}(A \vee B) \dashv\vdash {\sim}A \,\&\, {\sim}B$	(SDN)	${\sim}{\sim}A * {\sim}{\sim}B \dashv\vdash A * B$
(DEM)	${\sim}({\sim}A \vee {\sim}B) \dashv\vdash A \,\&\, B$	(SDN)	$A * B \dashv\vdash {\sim}{\sim}A * B \dashv\vdash A * {\sim}{\sim}B$
(DEM)	${\sim}({\sim}A \,\&\, {\sim}B) \dashv\vdash A \vee B$	(LEM)	$\vdash A \vee {\sim}A$
(QS)	$(\forall x){\sim}\phi x \dashv\vdash {\sim}(\exists x)\phi x$	(QS)	$(\exists x){\sim}\phi x \dashv\vdash {\sim}(\forall x)\phi x$
		(AV)	$\psi \dashv\vdash \psi'$

In (Com), '$*$' can be any binary connective *except* '$\to$'. In (SDN), '$*$' can be *any* binary connective. In (AV), ψ must be *closed*, and ψ' must be an *alphabetic variant* of ψ.

The Value of (QS) — Its Four Simplest Instances

(∀x)~Fx ⊢ ~(∃x)Fx

1	(1)	(∀x)~Fx	Premise
2	(2)	(∃x)Fx	Ass
3	(3)	Fa	Ass
1	(4)	~Fa	1 ∀E
1,3	(5)	⋏	4,3 ~E
1,2	(6)	⋏	2,3,5 ∃E
1	(7)	~(∃x)Fx	2,6 ~I

~(∃x)Fx ⊢ (∀x)~Fx

1	(1)	~(∃x)Fx	Premise
2	(2)	Fa	Ass
2	(3)	(∃x)Fx	2 ∃I
1,2	(4)	⋏	1,3 ~E
1	(5)	~Fa	2,4 ~I
1	(6)	(∀x)~Fx	5 ∀I

(∃x)~Fx ⊢ ~(∀x)Fx

1	(1)	(∃x)~Fx	Premise
2	(2)	(∀x)Fx	Ass
3	(3)	~Fa	Ass
2	(4)	Fa	2 ∀E
2,3	(5)	⋏	3,4 ~E
1,2	(6)	⋏	1,3,5 ∃E
1	(7)	~(∀x)Fx	2,6 ~I

~(∀x)Fx ⊢ (∃x)~Fx

1	(1)	~(∀x)Fx	Premise
2	(2)	~(∃x)~Fx	Ass
3	(3)	~Fa	Ass
3	(4)	(∃x)~Fx	3 ∃I
2,3	(5)	⋏	2,4 ~E
2	(6)	~~Fa	3,5 ~I
2	(7)	Fa	6 DN
2	(8)	(∀x)Fx	7 ∀I
1,2	(9)	⋏	1,8 ~E
1	(10)	~~(∃x)~Fx	2,9 ~I
1	(11)	(∃x)~Fx	10 DN

Three Examples Involving the LMPL SI Extension (QS)

- Here are three examples of proofs involving SI (QS):

1. ${\sim}(\forall x){\sim}Fx \vdash (\exists x)Fx$ [p. 207, #7 ⇐]
2. ${\sim}(\exists x)(Fx \,\&\, Gx) \vee (\exists x){\sim}Gx, (\forall y)Gy \vdash (\forall z)(Fz \to {\sim}Gz)$ [p. 205, ex. 1]
3. $(\forall x)Fx \to A \vdash (\exists x)(Fx \to A)$ [p. 205, ex. 2]

Proof of (1)

1	(1)	$\sim(\forall x)\sim Fx$	Premise
2	(2)	$\sim(\exists x)Fx$	Assumption
2	(3)	$(\forall x)\sim Fx$	2 SI (QS)
1,2	(4)	$\curlywedge$	1, 3 ~E
1	(5)	$\sim\sim(\exists x)Fx$	2, 4 ~I
1	(6)	$(\exists x)Fx$	5 DN

Proof of (2)

1	(1)	$\sim(\exists x)(Fx \,\&\, Gx) \vee (\exists x)\sim Gx$	Premise
2	(2)	$(\forall y)Gy$	Premise
3	(3)	$\sim(\exists x)(Fx \,\&\, Gx)$	Assumption
3	(4)	$(\forall x)\sim(Fx \,\&\, Gx)$	3 SI (QS)
3	(5)	$\sim(Fa \,\&\, Ga)$	4 ∀E
3	(6)	$\sim Fa \vee \sim Ga$	5 SI (DeM)
3	(7)	$Fa \rightarrow \sim Ga$	6 SI (Imp)
3	(8)	$(\forall z)(Fz \rightarrow \sim Gz)$	7 ∀I
9	(9)	$(\exists x)\sim Gx$	Assumption
10	(10)	$\sim Ga$	Assumption
2	(11)	Ga	2 ∀E
2,10	(12)	$\curlywedge$	10, 11 ~E
2,10	(13)	$(\forall z)(Fz \rightarrow \sim Gz)$	12 SI (EFQ)
2,9	(14)	$(\forall z)(Fz \rightarrow \sim Gz)$	9, 10, 13 ∃E
1,2	(15)	$(\forall z)(Fz \rightarrow \sim Gz)$	1, 3, 8, 9, 14 ∨E

Proof of (3)

Problem is: $(\forall x)Fx \rightarrow A \vdash (\exists x)(Fx \rightarrow A)$

1	(1)	$(\forall x)Fx \rightarrow A$	Premise
1	(2)	$\sim(\forall x)Fx \vee A$	1 SI (Imp)
3	(3)	$\sim(\forall x)Fx$	Assumption
3	(4)	$(\exists x)\sim Fx$	3 SI (QS)
5	(5)	$\sim Fa$	Assumption
5	(6)	$Fa \rightarrow A$	5 SI (PMI)
5	(7)	$(\exists x)(Fx \rightarrow A)$	6 ∃I
3	(8)	$(\exists x)(Fx \rightarrow A)$	4,5,7 ∃E
9	(9)	A	Assumption
9	(10)	$Fa \rightarrow A$	9 SI (PMI)
9	(11)	$(\exists x)(Fx \rightarrow A)$	10 ∃I
1	(12)	$(\exists x)(Fx \rightarrow A)$	2,3,8,9,11 ∨E

The Value of (AV)

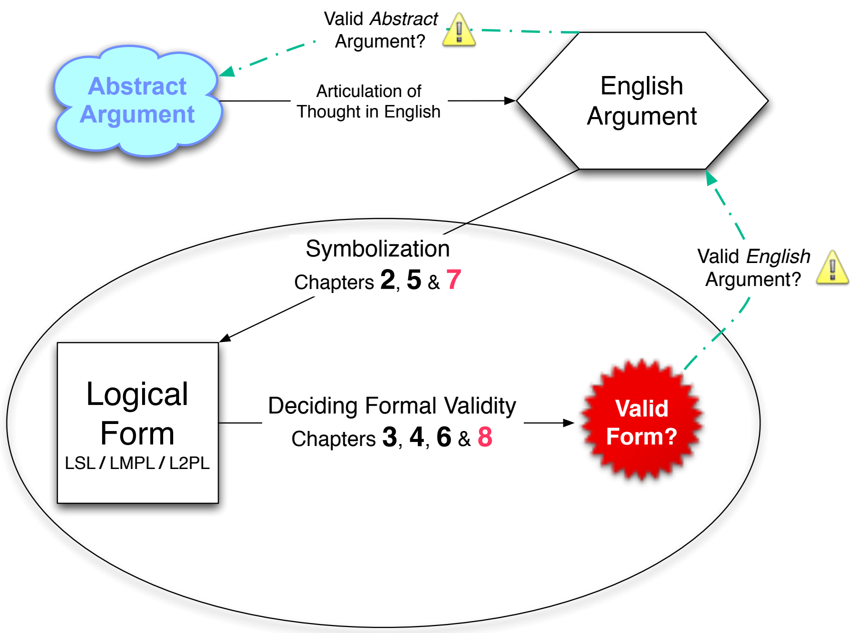
- Here are the two simplest instances of (AV):

$(\forall x)Fx \vdash (\forall y)Fy$				$(\exists x)Fx \vdash (\exists y)Fy$			
1	(1)	$(\forall x)Fx$	Premise	1	(1)	$(\exists x)Fx$	Premise
1	(2)	Fa	1 ∀E	2	(2)	Fa	Ass
1	(3)	$(\forall y)Fy$	2 ∀I	2	(3)	$(\exists y)Fy$	2 ∃I
				1	(4)	$(\exists y)Fy$	1,2,3 ∃E

- Here's an (AV)-aided proof of the following sequent

$$(\forall x)Fx, (\forall y)Fy \rightarrow (\forall y)Gy \vdash (\forall z)Gz$$

1	(1)	$(\forall x)Fx$	Premise
2	(2)	$(\forall y)Fy \rightarrow (\forall y)Gy$	Premise
1	(3)	$(\forall y)Fy$	1 SI (AV)
1,2	(4)	$(\forall y)Gy$	2,3 →E
1,2	(5)	$(\forall z)Gz$	4 SI (AV)



Beyond LMPL: 2-Place Predicates (*a.k.a.*, Relations) II

- From the point of view of logic (as opposed to mathematics) what matters is *capturing validities*. And, LMPL captures more than LSL.
- But, LMPL also has its own *logical* limitations. The problem: we can't capture some of the intuitively valid arguments involving *relations*.
- Consider the following argument, which involves a 2-place predicate:

(1) Brutus killed Caesar.

(2) ∴ Brutus killed someone and someone killed Caesar.

- If we were to symbolize this argument using monadic predicates, we would end-up with something like the following LMPL reconstruction:

(1′) Kb.

(2′) ∴ $(\exists x)Bx \,\&\, (\exists y)Ky$.

Where Kx: x killed Caesar, Bx: Brutus killed x, and b: Brutus.

- This argument is *not* valid in LMPL. But, the English argument *is* valid!

- The problem here is that "x killed y" is a *2-place* predicate (or *relation*).
- If we expand our language to include predicates that can take 2 arguments, then we can capture statements and arguments like these.
- In chapter 7, a more general language is introduced that allows n-place predicates, for any finite n. We will only discuss 2-place predicates.
- For instance, we can introduce the 2-place predicate Kxy: x killed y. With this relation in hand, we can express the above argument as:

(1⋆) Kbc.

(2⋆) ∴ $(\exists x)Kbx \,\&\, (\exists y)Kyc$.

- In 2-place predicate logic ("L2PL"), this argument *is* valid. So, this is a more accurate and faithful formalization of the English argument.
- We will (in chapter 8) discuss the semantics for 2-place predicate logic (L2PL). The natural deduction system for L2PL is *the same as* LMPL's!
- Before that, we will look at various complexities of L2PL *symbolization*.

Some Sample L2PL Symbolization Problems

1. Someone loves someone. [Lxy: x loves y]
   - First, work on the the quantifier with widest scope, then *work in*.
   - There exists an x such that x loves someone.

   (i) $(\exists x)$ x loves someone.

   - Now, work on expression within the scope of the quantifier in (i).

   (ii) x loves someone
     - there exists a y such that Lxy
     - $(\exists y)Lxy$

   - Plugging the symbolization of (ii) into (i) yields the **final product**:

$$(\exists x)(\exists y)Lxy$$

2. Everyone loves everyone.
   - For all x, x loves everyone.
   - $(\forall x)$ x loves everyone.
   - x loves everyone $\mapsto (\forall y)Lxy$
   - $(\forall x)(\forall y)Lxy$
3. Everyone loves someone.
   - For all x, x loves someone.
   - $(\forall x)$ x loves someone.
   - x loves someone $\mapsto (\exists y)Lxy$
   - $(\forall x)(\exists y)Lxy$
4. Someone loves everyone.
   - There exists an x such that x loves everyone.
   - $(\exists x)$ x loves everyone.
   - x loves everyone $\mapsto (\forall y)Lxy$
   - $(\exists x)(\forall y)Lxy$

5. Everyone is loved by by someone.
   - $(\forall x)$ x is loved by someone.
   - x is loved by someone $\mapsto (\exists y)Lyx$
   - $(\forall x)(\exists y)Lyx$
6. Someone is loved by everyone.
   - $(\exists x)$ x is loved by everyone.
   - $(\exists x)(\forall y)Lyx$
7. Everyone killed a killer. [Kxy: x killed y]
   - $(\forall x)$ x killed a killer.
   - $(\forall x)$ x killed someone who killed someone.
     - x killed someone who killed someone.
       - There exists a y such that x killed y and y killed someone.
       - $(\exists y)[Kxy \,\&\, y$ killed someone$]$
         - y killed someone $\mapsto (\exists z)Kyz$
   - $(\forall x)(\exists y)[Kxy \,\&\, (\exists z)Kyz]$

8. Caesar's assassins are friends of Brutus.
   [Axy: x is an assassin of y, Fxy: x is a friend of y]
   - All of Caesar's assassins are friends of Brutus.
   - $(\forall x)$ (if x is an assassin of Caesar, then x is a friend of Brutus).
   - $(\forall x)[Axc \to Fxb]$
9. There is no greatest prime number. [$\mathcal{D} = \mathbb{N}$, Px: x is prime, Gxy: $x \geq y$]
   - It is not the case that there exists a greatest prime number.
   - $\sim(\exists x)$ x is the greatest prime number.
     - x is the greatest prime number.
       - x is a prime number and x is $\geq$ all prime numbers.
       - Px and x is $\geq$ all prime numbers.
         - x is $\geq$ all prime numbers.
         - For all y, if y is a prime number then $x \geq y$.
         - $(\forall y)(Py \to Gxy)$
   - $\sim(\exists x)[Px \,\&\, (\forall y)(Py \to Gxy)]$

10. Some of Caesar's advisers who knew all of Caesar's assassins suspected none of them. [Axy: x is an assassin of y, Vxy: x is an adviser of y, Sxy: x suspected y, Kxy: x knew y, c: Caesar].
    - There exists an x such that x is an adviser of Caesar who knew all of Caesar's assassins and x suspected none of Caesar's assassins.
    - $(\exists x)$ (x is an adviser of Caesar who knew all of Caesar's assassins and x suspected none of Caesar's assassins)
    - $(\exists x)$ [(Vxc & x knew all of Caesar's assassins) & x suspected none of Caesar's assassins]
      - x knew all of Caesar's assassins
        - For all y, if y is an assassin of Caesar then x knew y.
        - $(\forall y)(Ayc \to Kxy)$
      - x suspected none of Caesar's assassins
        - For all y, if y is an assassin of Caesar then x did not suspect y.
        - $(\forall y)(Ayc \to \sim Sxy)$
    - $(\exists x)[(Vxc \,\&\, (\forall y)(Ayc \to Kxy)) \,\&\, (\forall y)(Ayc \to \sim Sxy)]$

11. Things which equal the same thing also equal one another. [Exy: $x = y$]
    - All things that are equal to the same thing are equal to each other.
    - For all x, y, z: If x and y are both equal to z, then x is equal to y.
    - $(\forall x)(\forall y)(\forall z)[(Exz \,\&\, Eyz) \rightarrow Exy]$

    Note: Any binary relation E that has this property is called a *Euclidean* relation (this is because the above statement is a quote from Euclid).

12. **An argument**. If there are any barbers then there is a barber who shaves all and only those who do not shave themselves. Therefore there are no barbers. [$\mathcal{D}$ = persons, Bx: x is a barber, Sxy: x shaves y].
    - Conclusion: There are no barbers. Easy: $\sim(\exists x)Bx$.
    - Premise: If there are any barbers then there is a barber who shaves all and only those who do not shave themselves.
      - Antecedent: there are (some) barbers. Easy: $(\exists x)Bx$.
      - Consequent: there is a barber who shaves all and only those who do not shave themselves.

  * There exists an x such that x is a barber and x shaves all and only those who do not shave themselves.
  * $(\exists x)$ (Bx & x shaves all and only those who do not shave themselves)
  * x shaves all and only those who do not shave themselves.
  * All those who do not shave themselves are shaved by x, and only those who do not shave themselves are shaved by x.
  * $(\forall y)(\sim Syy \rightarrow Sxy) \,\&\, (\forall y)(Sxy \rightarrow \sim Syy)$
  * That is, $(\forall y)(\sim Syy \leftrightarrow Sxy)$.
- Consequent: $(\exists x)[Bx \,\&\, (\forall y)(\sim Syy \leftrightarrow Sxy)]$.

- Premise: $(\exists x)Bx \rightarrow (\exists x)[Bx \,\&\, (\forall y)(\sim Syy \leftrightarrow Sxy)]$.

Entire Argument (which is a valid L2PL argument, by the way!):

$(\exists x)Bx \rightarrow (\exists x)[Bx \,\&\, (\forall y)(\sim Syy \leftrightarrow Sxy)]$

$\therefore \sim(\exists x)Bx$

Four Important Properties of Binary Relations

- **Reflexivity**. A binary relation R is said to be *reflexive* iff $(\forall x)Rxx$.
- **Symmetry**. R is *symmetric* iff $(\forall x)(\forall y)(Rxy \rightarrow Ryx)$.
- **Transitivity**. R is *transitive* iff $(\forall x)(\forall y)(\forall z)[(Rxy \,\&\, Ryz) \rightarrow Rxz]$.
- If R has *all three* of these properties, then R is an *equivalence relation*.
- **Fact**. If R is Euclidean and reflexive, then R is an equivalence relation.

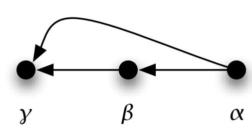
Relation	Reflexive?	Symmetric?	Transitive?	Euclidean?
$x > y$	No	No	Yes	No
$x \geq y$	Yes	No	Yes	No
x is a sibling of y	No	Yes	No	No
$x \approx y$	Yes	Yes	No	No
x respects y	No	No	No	No
$x = y$	Yes	Yes	Yes	Yes

L2PL Interpretations I

- Here's an example L2PL interpretation. Oxy: x was older than y, $\mathcal{D}$: The Three Stooges, Ref(a) = Curly, Ref(b) = Larry, and Ref(c) = Moe.
- The matrix representation of Ext(O) for this interpretation is:

O	α	β	γ
α	−	+	+
β	−	−	+
γ	−	−	−

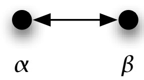
- The pictorial or diagrammatic representation of Ext(O) is:



L2PL Interpretations III

($\mathcal{I}_1$) Let $\mathcal{D}$ be the set consisting of George W. Bush (α) and Jeb Bush (β). And, let Bxy: x is a brother of y. Determine $\mathcal{I}_1$-truth-values for:

1. $(\forall x)(\exists y)Bxy$
2. $(\exists y)(\forall x)Bxy$



- (1) is $\top$ on $\mathcal{I}_1$, since *both* of its $\mathcal{D}$-instances are $\top$ on $\mathcal{I}_1$.
  - '$(\exists y)Bay$' is $\top$ on $\mathcal{I}_1$ because its instance 'Bab' is $\top$ on $\mathcal{I}_1$.
    - That is, $\langle \alpha, \beta \rangle \in \text{Ext}(B)$. Note: $\text{Ext}(B) = \{\langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle\}$.
  - '$(\exists y)Bby$' is $\top$ on $\mathcal{I}_1$ because its instance 'Bba' is $\top$ on $\mathcal{I}_1$.
- (2) is $\bot$ on $\mathcal{I}_1$, since *both* of its $\mathcal{D}$-instances are $\bot$ on $\mathcal{I}_1$.
  - '$(\forall x)Bxa$' is $\bot$ on $\mathcal{I}_1$ because its instance 'Baa' is $\bot$ on $\mathcal{I}_1$.
    - That is, $\langle \alpha, \alpha \rangle \notin \text{Ext}(B)$.
  - '$(\forall x)Bxb$' is $\bot$ on $\mathcal{I}_1$ because its instance 'Bbb' is $\bot$ on $\mathcal{I}_1$.

L2PL Interpretations IV

- Just as with LMPL, L2PL interpretations can be used as counterexamples to validity claims. Establishing $\nvDash$ claims works just as you'd expect.
- We have just seen an L2PL interpretation that shows the following:

$$(\forall x)(\exists y)Rxy \nvDash (\exists x)(\forall y)Rxy$$

- Interpretation $\mathcal{I}_1$ on the previous slide is a counterexample. Why?
  - $(\forall x)(\exists y)Bxy$ is $\top$ on $\mathcal{I}_1$, since both of its instances are $\top$ on $\mathcal{I}_1$.
  - $(\exists x)(\forall y)Rxy$ is $\bot$ on $\mathcal{I}_1$, since both of its instances are $\bot$ on $\mathcal{I}_1$.
- Here is a *very important* L2PL invalidity:

(†) $(\forall x)(\exists y)Rxy, (\forall x)(\forall y)(\forall z)[(Rxy \,\&\, Ryz) \to Rxz] \nvDash (\exists x)Rxx$

- (†) reveals a surprising difference between LMPL (and LSL) and L2PL — **sometimes *infinite* interpretations are needed to prove $\nvDash$ in L2PL!**

Why (†) is So Important — L2PL *vs* LMPL: Infinite Domains

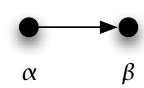
- In LMPL, if p is true on any interpretation $\mathcal{I}$, then it is true on a *finite* interpretation. Indeed, p will be true on an interpretation of size no greater than 2^k, where k is the # of monadic predicate letters in p.
- In L2PL, some statements are true *only* on *infinite* interpretations. It is for this reason that there is no general decision procedure for validity (or logical truth) in L2PL. (†) on the last slide is a good example of this.

(†) $(\forall x)(\exists y)Rxy, (\forall x)(\forall y)(\forall z)[(Rxy \,\&\, Ryz) \to Rxz] \nvDash (\exists x)Rxx$

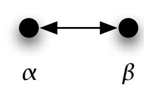
- **Fact**. *Only infinite interpretations $\mathcal{I}$ can be counterexamples to the validity in* (†). To see why, try to *construct* such an interpretation.
- We start by showing that no interpretation $\mathcal{I}_1$ with a 1-element domain can be an interpretation on which the premises of (†) are $\top$ and its conclusion is $\bot$. Then, we will repeat this argument for I_2 and $\mathcal{I}_3$.
- This reasoning can, in fact, be shown correct for *all* (finite) n. So, only $\mathcal{I}$'s with infinite domains will work [*e.g.*, $\mathcal{D} = \mathbb{N}$, Rxy: $x < y$].
- Begin with a 1-element domain $\{\alpha\}$. For the conclusion of (4) to be $\bot$, no

object can be related to itself: $(\forall x){\sim}Rxx$. Thus, we must have ${\sim}Raa$:

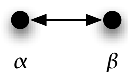
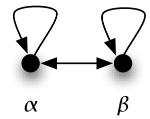
- But, to make the first premise $\top$, we need there to be *some* y such that Ray is $\top$. That means we need *another object* β to allow Rab. Thus:



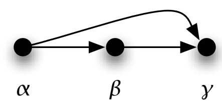
- Now, because we need the conclusion to remain $\bot$, we must have ${\sim}Rbb$. And, because we need the first premise to remain $\top$, we need there to be *some* y such that Rby is $\top$. We could *try* to make Rba $\top$, as follows:



- But, this picture is not consistent with the second premise being ⊤ and (at the same time) the conclusion being ⊥. If R is transitive, then $Rab \,\&\, Rba$ (as pictured) entails Raa, which makes the conclusion ⊤.

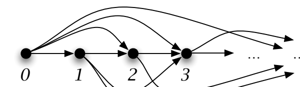
Transitivity of and  entails: 

- Thus, the only way to consistently ensure that there is some y such that Rby is to introduce *yet another object* γ (such that Rbc), which yields:



- Again, in order to make the conclusion ⊥, we must have $\sim Rcc$, and in order to make the first premise ⊤, there must be some y such that Rcy.
- We could *try* to make either Rca or Rcb true. But, both of these choices will end-up with the same sort of inconsistency we just saw with β.

- In other words, *no finite interpretation* will give us what we want here.
- However, if we let $\mathcal{D} = \mathbb{N}$ and Rxy: $x < y$, then we get what we want.



- That is, the relation Rxy: $x < y$ *on the natural numbers* $\mathbb{N}$ is such that:
  – For all x, there exists a y such that $x < y$. [seriality]
  – For all x, y, z, if $x < y$ and $y < z$, then $x < z$. [transitivity]
  – For all x, $x \not< x$. [irreflexivity]
- It is crucial that the set $\mathbb{N}$ of *all* natural numbers is *infinite*. The relation $<$ cannot satisfy all three of these properties on *any finite* domain.
- *I.e.*, no finite subset of $\mathbb{N}$ will suffice to show that the invalidity in (4) holds. Equivalently, the following sentence of L2PL is ⊥ on *all finite* $\mathcal{I}$'s:
  $p \quad (\forall x)(\exists y)Rxy \,\&\, (\forall x)(\forall y)(\forall z)[(Rxy \,\&\, Ryz) \rightarrow Rxz] \,\&\, (\forall x){\sim}Rxx$
- This sort of thing *cannot happen* in LMPL. In this sense, the introduction of a single 2-place predicate involves a *quantum leap* in complexity.

Some Further Remarks on Validity in L2PL

- As I just explained, there is no general decision procedure for ⊨ claims in L2PL. This is because we can't always establish ⊭ claims in finite time.
- However, there is a method for proving ⊨ claims — *natural deduction.* And, L2PL's natural deduction system *is exactly the same as LMPL's*!
- Before we get to proofs, however, I want to look at the alternating quantifier example that I said separates LMPL and L2PL.
- As we have seen, $(\forall x)(\exists y)Rxy \nvDash (\exists y)(\forall x)Rxy$. But, the converse entailment *does* hold. That is, $(\exists y)(\forall x)Rxy \vDash (\forall x)(\exists y)Rxy$.
- We will *prove — i.e., deduce —* $(\exists y)(\forall x)Rxy \vdash (\forall x)(\exists y)Rxy$ shortly.
- Before we do that, let's think about $(\exists y)(\forall x)Rxy \vDash (\forall x)(\exists y)Rxy$ using our definitions, and our informal method of thinking of ∀ as & and ∃ as ∨. This is interesting for both directions of the entailment.
- But, we need to be much more careful here than with LMPL!

- First, consider what $(\exists y)(\forall x)Rxy$ says on a domain of size n:
  $(\exists y)(\forall x)Rxy \approx_n (\forall x)Rxa \vee (\forall x)Rxb \vee \cdots \vee (\forall x)Rxn$
  $\approx_n (Raa \,\&\, \cdots \,\&\, Rna) \vee (Rab \,\&\, \cdots \,\&\, Rnb) \vee \cdots \vee (Ran \,\&\, \cdots \,\&\, Rnn)$
- Next, consider what $(\forall x)(\exists y)Rxy$ says on a domain of size n:
  $(\forall x)(\exists y)Rxy \approx_n (\exists y)Ray \,\&\, (\exists y)Rby \,\&\, \cdots \,\&\, (\exists y)Rny$
  $\approx_n (Raa \vee \cdots \vee Ran) \,\&\, (Rba \vee \cdots \vee Rbn) \,\&\, \cdots \,\&\, (Rna \vee \cdots \vee Rnn)$
- Then, we notice that these two sentential forms are intimately related. Specifically, we note that $(\exists y)(\forall x)Rxy$ has the following n-form:
  $\mathcal{X}_n = (p_1 \,\&\, p_2 \,\&\, \cdots \,\&\, p_n) \vee (q_1 \,\&\, q_2 \,\&\, \cdots \,\&\, q_n) \vee \cdots \vee (r_1 \,\&\, r_2 \,\&\, \cdots \,\&\, r_n)$
- And, we notice that $(\forall x)(\exists y)Rxy$ has the following n-form:
  $\mathcal{Y}_n = (p_1 \vee q_1 \vee \cdots \vee r_1) \,\&\, (p_2 \vee q_2 \vee \cdots \vee r_2) \,\&\, \cdots \,\&\, (p_n \vee q_n \vee \cdots \vee r_n)$
- **Fact**. $\mathcal{X}_n \vDash \mathcal{Y}_n$, *for any* n. Each disjunct of $\mathcal{X}_n$ entails every conjunct of $\mathcal{Y}_n$. **Caution!** This ***doesn't*** show that $(\exists y)(\forall x)Rxy \vDash (\forall x)(\exists y)Rxy$!
- **Fact**. $\mathcal{Y}_n \nvDash \mathcal{X}_n$, for all $n > 1$. This can be shown (next slide) using only LSL reasoning. This ***does*** show that $(\forall x)(\exists y)Rxy \nvDash (\exists y)(\forall x)Rxy$.

- The moral is that our "informal" semantical approach to the quantifiers works for LMPL, since no infinite domains are required for $\nvDash$ in LMPL.
- However, our "informal" semantical approach breaks down for L2PL, since we sometimes need an infinite domain to establish $\nvDash$ in L2PL.
- In L2PL, if the "informal" method above reveals $p_n \nvDash q_n$ for *some* finite n, then it *does* follow that $p \nvDash q$. For instance, $\mathcal{Y}_2 \nvDash \mathcal{X}_2$ on the last slide:
  - $(Raa \vee Rab) \,\&\, (Rba \vee Rbb) \nvDash (Raa \,\&\, Rba) \vee (Rab \,\&\, Rbb)$
  - This is just an LSL problem with 4-atoms [$A = Raa$, $B = Rab$, $C = Rba$, $D = Rbb$]. Truth-tables will generate a counterexample.
- On the other hand, if (in L2PL) our "informal" method indicates (as above) that $p_n \vDash q_n$ for *all* finite n, this does *not* guarantee $p \vDash q$. *E.g.*:
  - $p = (\forall x)(\exists y)Rxy \,\&\, (\forall x)(\forall y)(\forall z)[(Rxy \,\&\, Ryz) \to Rxz]$.
  - $q = (\exists x)Rxx$.
- We showed above (informally) that $p_n \vDash q_n$ for *all* finite n. But, we also saw that there are infinite interpretations on which p is $\top$ but q is $\bot$.