

Announcements and Such

- Today's Music: *Neil Young*
 - HW #4 is due on Thursday @ 4pm, usual drill (chapter 4 — proofs).
 - I've posted my solutions for HW #2 and HW #3.
 - Grade Curve (so far). Take the average of:
 - (1) your average HW score (all on 100-point scale), and
 - (2) your mid-term score.
 - The approximate "curve" for the course is as follows:
A-ish (≥ 90), B-ish (80-90), C-ish (70-80), D-ish (60-70).
 - This should be a reasonably good (but not perfect) guide to where the (overall) grade curve will end-up for the entire course.
 - Today: Chapter 4, Continued
 - More on \sim and \vee rules. Sequent & Theorem Introduction (SI/TI).
- ☞ **You should be doing as many proofs as you can.**

The Three Negation Rules

| Negation Elimination ($\sim E$) | | | Negation Introduction ($\sim I$) | | |
|------------------------------------|----------|------------------------|------------------------------------|----------|------------------------|
| a_1, \dots, a_n | (j) | $\sim q$ | j | (j) | p Assumption |
| | \vdots | | | \vdots | |
| b_1, \dots, b_u | (k) | q | a_1, \dots, a_n | (k) | \wedge |
| | \vdots | | | \vdots | |
| $a_1, \dots, a_n, b_1, \dots, b_u$ | (m) | \wedge j, k $\sim E$ | $\{a_1, \dots, a_n\}/j$ | (m) | $\sim p$ j, k $\sim I$ |

Double Negation (DN)

| | | |
|-------------------|-----|---------------|
| a_1, \dots, a_n | (j) | $\sim \sim p$ |
| a_1, \dots, a_n | (k) | p j DN |

The Two Disjunction Rules

| | | | | | | | |
|-------------------|----------|------------|------------|-------------------|-------------------|-----|-----------------------|
| a_1, \dots, a_n | (j) | p | | a_1, \dots, a_n | (j) | q | |
| | \vdots | | | | \vdots | | |
| a_1, \dots, a_n | (k) | $p \vee q$ | j $\vee I$ | OR | a_1, \dots, a_n | (k) | $p \vee q$ j $\vee I$ |

| | | |
|-------------------|----------|----------------------------|
| a_1, \dots, a_n | (g) | $p \vee q$ |
| | \vdots | |
| h | (h) | p Assumption |
| | \vdots | |
| b_1, \dots, b_u | (i) | r |
| | \vdots | |
| j | (j) | q Assumption |
| | \vdots | |
| c_1, \dots, c_w | (k) | r |
| | \vdots | |
| \mathcal{A} | (m) | r g, h, i, j, k $\vee E$ |

$$[\mathcal{A} = \{a_1, \dots, a_n\} \cup \{b_1, \dots, b_u\}/h \cup \{c_1, \dots, c_w\}/j]$$

A Simple Example Involving $\vee I$ and $\vee E$

- Here's a proof of the sequent: $A \vee B \vdash B \vee A$.

Problem is: $A \vee B \vdash B \vee A$

| | | | |
|---|-----|------------|-------------------------|
| 1 | (1) | $A \vee B$ | Premise |
| 2 | (2) | A | Assumption ($\vee E$) |
| 2 | (3) | $B \vee A$ | 2 $\vee I$ |
| 4 | (4) | B | Assumption ($\vee E$) |
| 4 | (5) | $B \vee A$ | 4 $\vee I$ |
| 1 | (6) | $B \vee A$ | 1,2,3,4,5 $\vee E$ |

An Example Involving \vee E and DN

- Here's a proof of the sequent: $A \vee B, \sim B \vdash A$.

Problem is: $A \vee B, \sim B \vdash A$

| | | |
|-------|------------------|---------------------------|
| 1 | (1) $A \vee B$ | Premise |
| 2 | (2) $\sim B$ | Premise |
| 3 | (3) $\sim A$ | Assumption (for \sim I) |
| 4 | (4) A | Assumption (for \vee E) |
| 3,4 | (5) Δ | 3,4 \sim E |
| 6 | (6) B | Assumption (for \vee E) |
| 2,6 | (7) Δ | 2,6 \sim E |
| 1,2,3 | (8) Δ | 1,4,5,6,7 \vee E |
| 1,2 | (9) $\sim\sim A$ | 3,8 \sim I |
| 1,2 | (10) A | 9 DN |

A Tricky Example Involving \vee I and All 3 Negation Rules

- Here's a proof of the *theorem*: $\vdash A \vee \sim A$.

Problem is: $\vdash A \vee \sim A$

| | | |
|-----|-------------------------------|------------------------|
| 1 | (1) $\sim(A \vee \sim A)$ | Assumption (\sim I) |
| 2 | (2) A | Assumption (\sim I) |
| 2 | (3) $A \vee \sim A$ | 2 \vee I |
| 1,2 | (4) Δ | 1,3 \sim E |
| 1 | (5) $\sim A$ | 2,4 \sim I |
| 1 | (6) $A \vee \sim A$ | 5 \vee I |
| 1 | (7) Δ | 1,6 \sim E |
| | (8) $\sim\sim(A \vee \sim A)$ | 1,7 \sim I |
| | (9) $A \vee \sim A$ | 8 DN |

A Third Example Involving \vee I and \vee E

- Here's a proof of the sequent: $A \vee (B \& C) \vdash (A \vee B) \& (A \vee C)$.

| | | |
|---|---------------------------------|------------------------|
| 1 | (1) $A \vee (B \& C)$ | Premise |
| 2 | (2) A | Assumption (\vee E) |
| 2 | (3) $A \vee B$ | 2 \vee I |
| 2 | (4) $A \vee C$ | 2 \vee I |
| 2 | (5) $(A \vee B) \& (A \vee C)$ | 3,4 $\&$ I |
| 6 | (6) $B \& C$ | Assumption (\vee E) |
| 6 | (7) B | 6 $\&$ E |
| 6 | (8) $A \vee B$ | 7 \vee I |
| 6 | (9) C | 6 $\&$ E |
| 6 | (10) $A \vee C$ | 9 \vee I |
| 6 | (11) $(A \vee B) \& (A \vee C)$ | 8,10 $\&$ I |
| 1 | (12) $(A \vee B) \& (A \vee C)$ | 1,2,5,6,11 \vee E |

Another Example Involving \vee

- Let's do a proof of: $(A \& B) \vee (A \& C) \vdash A \& (B \vee C)$

| | | |
|---|------------------------------|---------------------|
| 1 | (1) $(A \& B) \vee (A \& C)$ | Premise |
| 2 | (2) $A \& B$ | Ass (\vee E) |
| 2 | (3) A | 2 $\&$ E |
| 4 | (4) $A \& C$ | Ass (\vee E) |
| 4 | (5) A | 4 $\&$ E |
| 1 | (6) A | 1,2,3,4,5 \vee E |
| 2 | (7) B | 2 $\&$ E |
| 2 | (8) $B \vee C$ | 7 \vee I |
| 4 | (9) C | 4 $\&$ E |
| 4 | (10) $B \vee C$ | 9 \vee I |
| 1 | (11) $B \vee C$ | 1,2,8,4,10 \vee E |
| 1 | (12) $A \& (B \vee C)$ | 6,11 $\&$ I |

A Final Example Involving \vee and \sim

- Let's do a proof of: $\sim A \vee B \vdash A \rightarrow B$

Problem is : $\sim A \vee B \vdash A \rightarrow B$

| | | | |
|-----|------|-------------------|-------------------------------|
| 1 | (1) | $\sim A \vee B$ | Premise |
| 2 | (2) | A | Assumption (\rightarrow I) |
| 3 | (3) | $\sim A$ | Assumption (\vee E) |
| 4 | (4) | $\sim B$ | Assumption (\sim I) |
| 2,3 | (5) | \wedge | 3,2 \sim E |
| 2,3 | (6) | $\sim \sim B$ | 4,5 \sim I |
| 2,3 | (7) | B | 6 DN |
| 8 | (8) | B | Assumption (\vee E) |
| 1,2 | (9) | B | 1,3,7,8,8 \vee E |
| 1 | (10) | $A \rightarrow B$ | 2,9 \rightarrow I |

General Tips on Proof Strategy and Planning

- As a first line of attack, always try to prove your conclusion by using the introduction rule for its main connective as the main strategy.
- This will indicate what assumptions, if any, need to be made and what other formulae will need to be derived. This is "working backward".
- If these other formulae also contain connectives, then try to prove them by introducing their main connectives. Work backward, as far as possible.
- When this technique can no longer be applied, inspect your current stock of premises and assumptions to see if they have any *obvious* consequences.
- If your current premises and assumption contain a disjunction ' $r \vee s$ ', see if you can prove your current goal formula p from *each* of its disjuncts r and s (using your current premises and assumptions). If you think you can, then try using \vee E to prove p . If no disjunction appears anywhere in your current of premises/assumptions, then \vee E is probably not a good strategy.
- If you have tried everything you can think of to prove your current goal p , try assuming ' $\sim p$ ' and aim for ' $\sim \sim p$ ' by \sim E, \sim I; then use DN.

When to Make Assumptions, and When Not to

- In constructing a proof, any assumptions you make must eventually be discharged, so you should only make assumptions in connection with the three rules which discharge assumptions.
- In other words, if you make an assumption p in a proof, you *must* be able to give one of the following three reasons:
 - p is the antecedent of a conditional ' $p \rightarrow q$ ' you are trying to derive using the \rightarrow I rule (then, try to prove q).
 - You are trying to derive ' $\sim p$ ', so you assume p with an eye toward using the \sim I rule (then, try to prove \wedge).
 - p is one of the disjuncts of a disjunction ' $p \vee q$ ' (*somewhere in your current stock of premises and assumptions!*) to which you will be applying \vee E (then, try to prove some r from each).
- Remember, only the three rules \rightarrow I, \sim I, and \vee E involve making assumptions. *No other rules can discharge assumptions.*

10 More Examples Involving \vee I and \vee E

- $(A \& B) \vee (A \& C) \vdash A$ [p. 111, ex. 2]
- $(A \rightarrow \wedge) \vee (B \rightarrow \wedge), B \vdash \sim A$ [p. 116, §4.5, ex. 11]
- $(A \vee B) \vee C \vdash A \vee (B \vee C)$ [p. 116, ex. 19]
- $A \vee B \vdash (A \rightarrow B) \rightarrow B$ [p. 116, ex. 10]
- $A \& B \vdash \sim(\sim A \vee \sim B)$ [p. 116, ex. 14 (\rightarrow)]
- $A \vee B \vdash \sim(\sim A \& \sim B)$ [p. 116, ex. 13]
- $\sim(A \& B) \vdash \sim A \vee \sim B$ [p. 116, ex. 16 (\rightarrow)]
- $\sim C \vee (A \rightarrow B) \vdash (C \& A) \rightarrow B$ [not in text]
- $\vdash (A \rightarrow B) \vee (B \rightarrow A)$ [not in text]
- $\sim(A \vee B) \vdash \sim A \& \sim B$ [not in text]

Proof of Example #1

Problem is: $(A \& B) \vee (A \& C) \vdash A$

| | | |
|---|------------------------------|-------------------------|
| 1 | (1) $(A \& B) \vee (A \& C)$ | Premise |
| 2 | (2) $A \& B$ | Assumption ($\vee E$) |
| 2 | (3) A | 2 &E |
| 4 | (4) $A \& C$ | Assumption ($\vee E$) |
| 4 | (5) A | 4 &E |
| 1 | (6) A | 1,2,3,4,5 $\vee E$ |

Proof of Example #2

Problem is: $(A \rightarrow \Delta) \vee (B \rightarrow \Delta), B \vdash \sim A$

| | | |
|-------|--|-------------------------|
| 1 | (1) $(A \rightarrow \Delta) \vee (B \rightarrow \Delta)$ | Premise |
| 2 | (2) B | Premise |
| 3 | (3) A | Assumption ($\sim I$) |
| 4 | (4) $A \rightarrow \Delta$ | Assumption ($\vee E$) |
| 3,4 | (5) Δ | 4,3 $\rightarrow E$ |
| 6 | (6) $B \rightarrow \Delta$ | Assumption ($\vee E$) |
| 2,6 | (7) Δ | 6,2 $\rightarrow E$ |
| 1,2,3 | (8) Δ | 1,4,5,6,7 $\vee E$ |
| 1,2 | (9) $\sim A$ | 3,8 $\sim I$ |

Proof of Example #3

Problem is: $(A \vee B) \vee C \vdash A \vee (B \vee C)$

| | | |
|---|--------------------------|-------------------------|
| 1 | (1) $(A \vee B) \vee C$ | Premise |
| 2 | (2) $A \vee B$ | Assumption ($\vee E$) |
| 3 | (3) A | Assumption ($\vee E$) |
| 3 | (4) $A \vee (B \vee C)$ | 3 $\vee I$ |
| 5 | (5) B | Assumption ($\vee E$) |
| 5 | (6) $B \vee C$ | 5 $\vee I$ |
| 5 | (7) $A \vee (B \vee C)$ | 6 $\vee I$ |
| 2 | (8) $A \vee (B \vee C)$ | 2,3,4,5,7 $\vee E$ |
| 9 | (9) C | Assumption ($\vee E$) |
| 9 | (10) $B \vee C$ | 9 $\vee I$ |
| 9 | (11) $A \vee (B \vee C)$ | 10 $\vee I$ |
| 1 | (12) $A \vee (B \vee C)$ | 1,2,8,9,11 $\vee E$ |

Proof of Example #4

Problem is: $A \vee B \vdash (A \rightarrow B) \rightarrow B$

| | | |
|-----|---------------------------------------|-------------------------|
| 1 | (1) $A \vee B$ | Premise |
| 2 | (2) $A \rightarrow B$ | Ass ($\rightarrow I$) |
| 3 | (3) A | Ass ($\vee E$) |
| 2,3 | (4) B | 2,3 $\rightarrow E$ |
| 5 | (5) B | Ass ($\vee E$) |
| 1,2 | (6) B | 1,3,4,5,5 $\vee E$ |
| 1 | (7) $(A \rightarrow B) \rightarrow B$ | 2,6 $\rightarrow I$ |

Proof of Example #5

Problem is: $A \& B \vdash \sim(\sim A \vee \sim B)$

| | | |
|-----|---------------------------------|-------------------------|
| 1 | (1) $A \& B$ | Premise |
| 2 | (2) $\sim A \vee \sim B$ | Assumption ($\sim I$) |
| 3 | (3) $\sim A$ | Assumption ($\vee E$) |
| 1 | (4) A | 1 &E |
| 1,3 | (5) Δ | 3,4 $\sim E$ |
| 6 | (6) $\sim B$ | Assumption ($\vee E$) |
| 1 | (7) B | 1 &E |
| 1,6 | (8) Δ | 6,7 $\sim E$ |
| 1,2 | (9) Δ | 2,3,5,6,8 $\vee E$ |
| 1 | (10) $\sim(\sim A \vee \sim B)$ | 2,9 $\sim I$ |

Proof of Example #6

Problem is : $A \vee B \vdash \sim(\sim A \& \sim B)$

| | | |
|-----|-------------------------------|--------------------|
| 1 | (1) $A \vee B$ | Premise |
| 2 | (2) $\sim A \& \sim B$ | Ass ($\sim I$) |
| 3 | (3) A | Ass ($\vee E$) |
| 2 | (4) $\sim A$ | 2 &E |
| 2,3 | (5) Δ | 4,3 $\sim E$ |
| 6 | (6) B | Ass ($\vee E$) |
| 2 | (7) $\sim B$ | 2 &E |
| 2,6 | (8) Δ | 7,6 $\sim E$ |
| 1,2 | (9) Δ | 1,3,5,6,8 $\vee E$ |
| 1 | (10) $\sim(\sim A \& \sim B)$ | 2,9 $\sim I$ |

Proof of Example #7

Problem is: $\sim(A \& B) \vdash \sim A \vee \sim B$

| | | |
|-----|--------------------------------------|-------------------------|
| 1 | (1) $\sim(A \& B)$ | Premise |
| 2 | (2) $\sim(\sim A \vee \sim B)$ | Assumption ($\sim I$) |
| 3 | (3) $\sim A$ | Assumption ($\sim I$) |
| 3 | (4) $\sim A \vee \sim B$ | 3 $\vee I$ |
| 2,3 | (5) Δ | 2,4 $\sim E$ |
| 2 | (6) $\sim \sim A$ | 3,5 $\sim I$ |
| 2 | (7) A | 6 DN |
| 8 | (8) $\sim B$ | Assumption ($\sim I$) |
| 8 | (9) $\sim A \vee \sim B$ | 8 $\vee I$ |
| 2,8 | (10) Δ | 2,9 $\sim E$ |
| 2 | (11) $\sim \sim B$ | 8,10 $\sim I$ |
| 2 | (12) B | 11 DN |
| 2 | (13) $A \& B$ | 7,12 &I |
| 1,2 | (14) Δ | 1,13 $\sim E$ |
| 1 | (15) $\sim \sim(\sim A \vee \sim B)$ | 2,14 $\sim I$ |
| 1 | (16) $\sim A \vee \sim B$ | 15 DN |

Proof of Example #8

Problem is: $\sim C \vee (A \rightarrow B) \vdash (C \& A) \rightarrow B$

| | | |
|-------|-------------------------------------|--------------------------------|
| 1 | (1) $\sim C \vee (A \rightarrow B)$ | Premise |
| 2 | (2) $C \& A$ | Assumption ($\rightarrow I$) |
| 3 | (3) $\sim B$ | Assumption ($\sim I$) |
| 4 | (4) $\sim C$ | Assumption ($\vee E$) |
| 2 | (5) C | 2 &E |
| 2,4 | (6) Δ | 4,5 $\sim E$ |
| 7 | (7) $A \rightarrow B$ | Assumption ($\vee E$) |
| 2 | (8) A | 2 &E |
| 2,7 | (9) B | 7,8 $\rightarrow E$ |
| 2,3,7 | (10) Δ | 3,9 $\sim E$ |
| 1,2,3 | (11) Δ | 1,4,6,7,10 $\vee E$ |
| 1,2 | (12) $\sim \sim B$ | 3,11 $\sim I$ |
| 1,2 | (13) B | 12 DN |
| 1 | (14) $(C \& A) \rightarrow B$ | 2,13 $\rightarrow I$ |

Proof of Example #9

Problem is: $\vdash (A \rightarrow B) \vee (B \rightarrow A)$

| | | | |
|-----|------|--|--------------------------------|
| 1 | (1) | $\neg((A \rightarrow B) \vee (B \rightarrow A))$ | Assumption (\neg -I) |
| 2 | (2) | B | Assumption (\rightarrow -I) |
| 3 | (3) | $\neg A$ | Assumption (\neg -I) |
| 4 | (4) | A | Assumption (\rightarrow -I) |
| 2 | (5) | $A \rightarrow B$ | 4,2 \rightarrow -I |
| 2 | (6) | $(A \rightarrow B) \vee (B \rightarrow A)$ | 5 \vee -I |
| 1,2 | (7) | Δ | 1,6 \neg -E |
| 1,2 | (8) | $\neg\neg A$ | 3,7 \neg -I |
| 1,2 | (9) | A | 8 DN |
| 1 | (10) | $B \rightarrow A$ | 2,9 \rightarrow -I |
| 1 | (11) | $(A \rightarrow B) \vee (B \rightarrow A)$ | 10 \vee -I |
| 1 | (12) | Δ | 1,11 \neg -E |
| 1 | (13) | $\neg\neg((A \rightarrow B) \vee (B \rightarrow A))$ | 1,12 \neg -I |
| 1 | (14) | $(A \rightarrow B) \vee (B \rightarrow A)$ | 13 DN |

Proof of Example #10

Problem is: $\neg(A \vee B) \vdash \neg A \& \neg B$

| | | | |
|-----|------|--------------------|------------------|
| 1 | (1) | $\neg(A \vee B)$ | Premise |
| 2 | (2) | A | Ass (\neg -I) |
| 2 | (3) | $A \vee B$ | 2 \vee -I |
| 1,2 | (4) | Δ | 1,3 \neg -E |
| 1 | (5) | $\neg A$ | 2,4 \neg -I |
| 6 | (6) | B | Ass (\neg -I) |
| 6 | (7) | $A \vee B$ | 6 \vee -I |
| 1,6 | (8) | Δ | 1,7 \neg -E |
| 1 | (9) | $\neg B$ | 6,8 \neg -I |
| 1 | (10) | $\neg A \& \neg B$ | 5,9 $\&$ I |

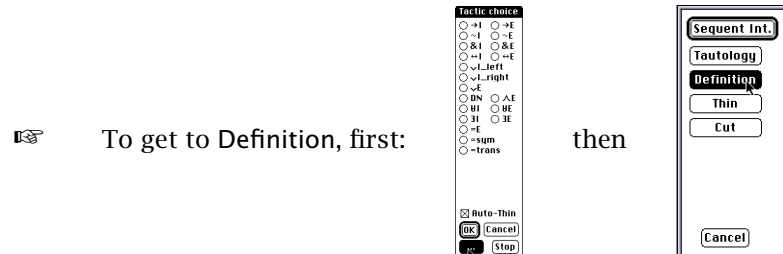
The Rule of Definition for the Biconditional

Rule of Definition for \leftrightarrow (Df): If ' $(p \rightarrow q) \& (q \rightarrow p)$ ' occurs as the entire formula at line j, then at line k we may write ' $p \leftrightarrow q$ ', labeling the line 'j Df' and writing on its left the same numbers as are on the left of j. Conversely, if ' $p \leftrightarrow q$ ' occurs as the entire formula at a line j, then at line k we may write ' $(p \rightarrow q) \& (q \rightarrow p)$ ', labeling the line 'j Df' and writing on its left the same numbers as are on the left of j.

| | | | |
|-------------------|----------|--|------|
| a_1, \dots, a_n | (j) | $(p \rightarrow q) \& (q \rightarrow p)$ | |
| | \vdots | | |
| a_1, \dots, a_n | (k) | $p \leftrightarrow q$ | j Df |
| | | OR | |
| a_1, \dots, a_n | (j) | $p \leftrightarrow q$ | |
| | \vdots | | |
| a_1, \dots, a_n | (k) | $(p \rightarrow q) \& (q \rightarrow p)$ | j Df |

Using \leftrightarrow in MacLogic

- Using the Definition strategy of MacLogic (accessed via the \dots button), we can implement our Df. rule for \leftrightarrow . *Do not use \leftrightarrow -I or \leftrightarrow -E!*
- Using MacLogic's Definition strategy is much simpler than using its Tautology strategy (I did that last time, which was cumbersome).



- Here is a non-trivial example: $A \leftrightarrow \neg B \vdash \neg(A \leftrightarrow B)$. Let's try to tackle this one, using MacLogic's Definition strategy for our Df.
- The shortest proof I've been able to find is 18 steps (next slide). Forbes gives a 20-stepper in his discussion of this example (p. 118).

Problem is : $A \leftrightarrow \sim B \vdash \sim(A \leftrightarrow B)$

| | | |
|-------|--|-----------------------|
| 1 | (1) $A \leftrightarrow \sim B$ | Ass |
| 2 | (2) $A \leftrightarrow B$ | Ass |
| 1 | (3) $(A \rightarrow \sim B) \& (\sim B \rightarrow A)$ | 1 Defn. |
| 1 | (4) $A \rightarrow \sim B$ | 3 &E |
| 1 | (5) $\sim B \rightarrow A$ | 3 &E |
| 6 | (6) B | Ass |
| 2 | (7) $(A \rightarrow B) \& (B \rightarrow A)$ | 2 Defn. |
| 2 | (8) $B \rightarrow A$ | 7 &E |
| 2,6 | (9) A | 8,6 \rightarrow E |
| 1,2,6 | (10) $\sim B$ | 4,9 \rightarrow E |
| 1,2,6 | (11) Δ | 10,6 \rightarrow E |
| 1,2 | (12) $\sim B$ | 6,11 \rightarrow I |
| 1,2 | (13) A | 5,12 \rightarrow E |
| 1,2 | (14) $\sim B$ | 4,13 \rightarrow E |
| 2 | (15) $A \rightarrow B$ | 7 &E |
| 1,2 | (16) B | 15,13 \rightarrow E |
| 1,2 | (17) Δ | 14,16 \rightarrow E |
| 1 | (18) $\sim(A \leftrightarrow B)$ | 2,17 \rightarrow I |

Sequent and Theorem Introduction: I

- You may have noticed that certain important sequents or theorems tend to get proven over and over again in different problems.
- For instance, the sequent $X \vee Y, \sim X \vdash Y$ is a very useful thing to know, as are the sequents $X \rightarrow Y, \sim Y \vdash \sim X$, $\wedge \vdash X$, and many others.
- It would be nice if we had a rule that allowed us to say “OK, I’ve proven this sequent already, so I don’t have to prove it again here”.
- We have two such rules. They are called *Sequent Introduction* (SI) for sequents, and *Theorem Introduction* (TI) for theorems.
- SI and TI allow us to avoid having to re-solve certain sub-problems that we already know how to solve. This makes proofs shorter.
- We will have a fixed list of sequents and theorems that we’ll be allowed to use in conjunction with SI and TI.

Sequent and Theorem Introduction: II

- Forbes lists a bunch of sequents and Theorems on page 123 that we may use with SI or TI. There’s a MacLogic file containing all of them.
- Here are a few of the sequents and theorems that tend to be useful:

| | |
|--|--------|
| $p \vee q, \sim p \vdash q$; or; $p \vee q, \sim q \vdash p$ | (DS) |
| $p \rightarrow q, \sim q \vdash \sim p$ | (MT) |
| $p \vdash q \rightarrow p$; or; $\sim p \vdash p \rightarrow q$ | (PMI) |
| $\vdash p \vee \sim p$ | (LEM) |
| $\sim(p \& q) \sim p \vee \sim q$ | (DEM) |
| $\sim(p \vee q) \sim p \& \sim q$ | (DEM) |
| $\sim(\sim p \vee \sim q) p \& q$ | (DEM) |
| $\sim(\sim p \& \sim q) p \vee q$ | (DEM) |
| $\wedge \vdash p$ | (EFQ) |
| $p \& (q \vee r) (p \& q) \vee (p \& r)$ | (DIST) |

Sequent and Theorem Introduction: III

- Remember the proof for #9 above: $\vdash (A \rightarrow B) \vee (B \rightarrow A)$.

| | | |
|-----|--|-------------------------------|
| 1 | (1) $\sim((A \rightarrow B) \vee (B \rightarrow A))$ | Assumption (\sim I) |
| 2 | (2) B | Assumption (\rightarrow I) |
| 3 | (3) $\sim A$ | Assumption (\sim I) |
| 4 | (4) A | Assumption (\rightarrow I) |
| 2 | (5) $A \rightarrow B$ | 4,2 \rightarrow I |
| 2 | (6) $(A \rightarrow B) \vee (B \rightarrow A)$ | 5 \vee I |
| 1,2 | (7) Δ | 1,6 \rightarrow E |
| 1,2 | (8) $\sim \sim A$ | 3,7 \rightarrow I |
| 1,2 | (9) A | 8 DN |
| 1 | (10) $B \rightarrow A$ | 2,9 \rightarrow I |
| 1 | (11) $(A \rightarrow B) \vee (B \rightarrow A)$ | 10 \vee I |
| 1 | (12) Δ | 1,11 \rightarrow E |
| | (13) $\sim \sim((A \rightarrow B) \vee (B \rightarrow A))$ | 1,12 \rightarrow I |
| | (14) $(A \rightarrow B) \vee (B \rightarrow A)$ | 13 DN |

Sequent and Theorem Introduction: IV

- Using TI and SI, we can obtain the following much simpler proof:

| | | | |
|---|-----|--|-------------------------|
| | (1) | $A \vee \sim A$ | TI (LEM) |
| 2 | (2) | A | Assumption ($\vee E$) |
| 2 | (3) | $B \rightarrow A$ | 2 SI (PMI) |
| 2 | (4) | $(A \rightarrow B) \vee (B \rightarrow A)$ | 3 $\vee I$ |
| 5 | (5) | $\sim A$ | Assumption ($\vee E$) |
| 5 | (6) | $A \rightarrow B$ | 5 SI (PMI) |
| 5 | (7) | $(A \rightarrow B) \vee (B \rightarrow A)$ | 6 $\vee I$ |
| | (8) | $(A \rightarrow B) \vee (B \rightarrow A)$ | 1,2,4,5,7 $\vee E$ |

- Here, LEM is the theorem $\vdash A \vee \sim A$ (which we have already proven), and PMI stands for either of the sequents $\sim A \vdash A \rightarrow B$ (used at line 6), or $A \vdash B \rightarrow A$ (used at line 3), both of which we've proven.
- SI allows you to use (any substitution instance of) any sequent that you've already proven to make an inference at any stage of a proof.
- TI allows you to write down (any substitution instance of) any theorem that you have already proven at any stage of a proof.

The Formal Definitions of SI and TI

- Sequent Introduction (SI).** Suppose $r_1, \dots, r_n \vdash s$ is a *substitution-instance* of the sequent $p_1, \dots, p_n \vdash q$ which we have already proved, and that the formulae r_1, \dots, r_n occur at lines j_1, \dots, j_n in a proof. Then we may infer s at line k , labeling the line ' j_1, \dots, j_n SI (Identifier)' and writing on the left all numbers which appear on the left of lines j_1, \dots, j_n .
- Theorem Introduction (TI).** If $\vdash s$ is a *substitution-instance* of some theorem $\vdash q$ which we have already proved, we may introduce a new line k into a proof with the formula s at it and no numbers on its left, labeling the line 'TI (Identifier)'.
- 'Identifier' stands for the name of a sequent or theorem that has already been proven (e.g., MT, DS, PMI, LEM, etc). See Forbes's list.
- Note: TI is just a *special case* of SI (with $n = 0$).

SI and TI: A Relatively Easy Example

- Use SI/TI to find a "short" proof of: $\sim(A \rightarrow (B \vee C)) \vdash (B \vee C) \rightarrow A$.

Problem is : $\sim(A \rightarrow (B \vee C)) \vdash (B \vee C) \rightarrow A$

| | | | |
|---|-----|----------------------------------|---------------|
| 1 | (1) | $\sim(A \rightarrow (B \vee C))$ | Premise |
| 1 | (2) | $A \& \sim(B \vee C)$ | 1 SI Neg-Imp1 |
| 1 | (3) | A | 2 &E |
| 1 | (4) | $(B \vee C) \rightarrow A$ | 3 SI PMI1 |

SI and TI: A More Challenging Example

- Use SI/TI to find a "short" proof of: $A \rightarrow (B \vee C) \vdash (A \rightarrow B) \vee (A \rightarrow C)$.
Problem is : $A \rightarrow (B \vee C) \vdash (A \rightarrow B) \vee (A \rightarrow C)$

| | | | |
|----|------|--|-------------------------|
| 1 | (1) | $A \rightarrow (B \vee C)$ | Premise |
| 1 | (2) | $\sim A \vee (B \vee C)$ | 1 SI IMP1 |
| 3 | (3) | $\sim A$ | Assumption ($\vee E$) |
| 3 | (4) | $A \rightarrow B$ | 3 SI PMI2 |
| 3 | (5) | $(A \rightarrow B) \vee (A \rightarrow C)$ | 4 $\vee I_{left}$ |
| 6 | (6) | $B \vee C$ | Assumption ($\vee E$) |
| 7 | (7) | B | Assumption ($\vee E$) |
| 7 | (8) | $A \rightarrow B$ | 7 SI PMI1 |
| 7 | (9) | $(A \rightarrow B) \vee (A \rightarrow C)$ | 8 $\vee I_{left}$ |
| 10 | (10) | C | Assumption ($\vee E$) |
| 10 | (11) | $A \rightarrow C$ | 10 SI PMI1 |
| 10 | (12) | $(A \rightarrow B) \vee (A \rightarrow C)$ | 11 $\vee I_{right}$ |
| 6 | (13) | $(A \rightarrow B) \vee (A \rightarrow C)$ | 6,7,9,10,12 $\vee E$ |
| 1 | (14) | $(A \rightarrow B) \vee (A \rightarrow C)$ | 2,3,5,6,13 $\vee E$ |