

Gödel's Metatheorem (45.17) and the Strong Completeness Theorem for FOTs (46.2)

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Before getting to the salient proofs, it's important to understand Hunter's terminology "consistent set Γ of WFFs of a first order theory K ". For Hunter, Γ is a consistent set of WFFs of K iff there is no WFF A of K such that $\Gamma \vdash_K A$ and $\Gamma \vdash_K \sim A$. As a result, this definition of "consistent set of WFFs Γ of K " implies that K is itself a consistent first-order theory! That is, an inconsistent first order theory K does not have any consistent sets of WFFs on this definition. This sounds a bit odd, but it's crucial for the proofs below. In this handout, I will go through the proper proofs of 45.17 and 46.2. To this end, I will begin with the background ingredients of the proof of 45.17: metatheorem 45.16, and Lemmas 1 and 2.

45.16. If Γ is a consistent set of *closed* WFFs of a first order theory K , then Γ has a denumerable model.

Proof. Assume Γ is a consistent set of *closed* WFFs of a first order theory K . Then, by Hunter's definition (above), there is no WFF A of K such that $\Gamma \vdash_K A$ and $\Gamma \vdash_K \sim A$. Therefore, it follows that the first order theory $K + \Gamma$ is a consistent first order theory. If $K + \Gamma$ were inconsistent, then there would have to be a WFF A of $K + \Gamma$ such that both A and $\sim A$ were theorems of $K + \Gamma$. That would imply the existence of an A such that $\Gamma \vdash_K A$ and $\Gamma \vdash_K \sim A$, which contradicts Hunter's definition of "consistent set of WFFs Γ of K ." Since $K + \Gamma$ is a consistent first order theory, it must have a denumerable model [this is implied by theorems 45.10–45.14]. Thus, Γ itself has a denumerable model (Γ is a subset of the set of theorems of $K + \Gamma$). \square

Lemma 1 for 45.17. If Γ is a consistent set of WFFs of a first order theory K , then Γ is also a consistent set of WFFs of the first order theory K' , where K' is the first order theory one gets when one adds denumerably many new constant symbols with an effective enumeration $\langle c_1, \dots, c_n, \dots \rangle$ to K .

Proof. Assume Γ is a consistent set of WFFs of a first order theory K , and assume that K' is K with the new constant symbols $\langle c_1, \dots, c_n, \dots \rangle$ added to it. Now, assume, for *reductio*, that Γ is an inconsistent set of K' . Then, by definition, this means that there is a WFF B of K' such that $\Gamma \vdash_{K'} B$ and $\Gamma \vdash_{K'} \sim B$. So, since derivations are finite, there is a finite subset $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{K'} B$ and $\Delta \vdash_{K'} \sim B$. These derivations in K' can be converted into derivations *in* K , as follows. Let $X' = Xv_i/c_i$, where v_i is the i th variable in our enumeration that does not occur in either of the derivations $\Delta \vdash_{K'} B$ or $\Delta \vdash_{K'} \sim B$, and c_i is the i th constant symbol in our enumeration of new symbols added to K to yield K' . Then, $\Delta' = \Delta$, since $\Delta \subseteq \Gamma$ and Γ is a set of WFFs of K (and so do not contain any c_i 's). Moreover, $\Delta \vdash_{K'} B'$ and $\Delta \vdash_{K'} \sim B'$. Why? Think about $\Delta \vdash_{K'} B$ (parallel argument for $\sim B$). There are four possibilities: (i) B is a logical axiom of K' , (ii) B is a proper axiom of K' , (iii) B follows by MP from two WFFs in Δ , or (iv) $B \in \Delta$. If B is a logical axiom of K' [case (i)], then B' is a logical axiom of K . There are five sub-cases here (this is just like part I of 45.12):

- (i.1) B is a logical axiom by K1–K3. Then, B is an instance of a propositional axiom schemata. In this case, $B' = Bv_i/c_i$ is also an instance of a propositional axiom schemata, since merely substituting v_i for c_i in B cannot change the propositional logical form of any formula B .
- (i.2) B is a logical axiom by K4. So, B is of the form $\wedge uC \supset Ct/u$, where t is free for u in C . Thus, $B' = (\wedge uC \supset Ct/u)v_i/c_i = (\wedge uCv_i/c_i \supset (Ct/u)v_i/c_i) = (\wedge uCv_i/c_i \supset (Cv_i/c_i)t/u)$, where t is free for u in C . This is because v_i does not occur in B (hence, v_i does not occur in C , Ct/u , or t), and so $(Ct/u)v_i/c_i = (Cv_i/c_i)t/u$. Therefore, B' is of the form $\wedge uD \supset Dt/u$ (with $D = Cv_i/c_i$), where t is free for u in D . So, B' is also a logical axiom by K4.
- (i.3) B is a logical axiom by K5. So, B is of the form $C \supset \wedge uC$, where u does not occur free in C . Thus, $B' = (C \supset \wedge uC)v_i/c_i = (Cv_i/c_i \supset \wedge uCv_i/c_i)$. Therefore, B' is of the form $D \supset \wedge uD$ (with $D = Cv_i/c_i$), where u does not occur free in D . So, B' is also a logical axiom by K5.

- (i.4) B is a logical axiom by K6. So, B is of the form $\wedge u(C \supset D) \supset (\wedge uC \supset \wedge uD)$. As a result, $B' = (\wedge u(C \supset D) \supset (\wedge uC \supset \wedge uD))v_i/c_i = \wedge u(Cv_i/c_i \supset Dv_i/c_i) \supset (\wedge uCv_i/c_i \supset \wedge uDv_i/c_i)$, which is also a logical axiom by K6.
- (i.5) B is a logical axiom by K7. So, B is of the form $\wedge uC$, where C is a logical axiom by K1-K6. Thus, $B' = \wedge uCv_i/c_i$, where Cv_i/c_i is a logical axiom by the above six arguments (which show that if C is an axiom by K1-K6, then so is $C' = Cv_i/c_i$). Hence, by K7, B' is also a logical axiom.

In the other three cases (ii-iv), $B' = B$, and this is why $\Delta \vdash_{K'} B \Rightarrow \Delta \vdash_K B'$ in those cases:

- (ii) If B is a proper axiom of K' , then B' is a proper axiom of K . For, if B is a proper axiom of K' , then $B' = Bv_i/c_i = B$, since the new c_i 's cannot occur in any proper axiom of K (so "replacing" c_i with v_i in B does nothing to B). So, in this case, $\Delta \vdash_{K'} B \Rightarrow \Delta \vdash_K B'$.
- (iii) If B follows by MP from two WFFs in Δ , then so does B' , since (again) $B' = B$. This is because the WFFs in Δ (WFFs of K) do not contain any of the new c_i , and therefore anything that follows by MP from formulas in Δ cannot contain any of the new c_i either. So, $B' = B$, and $\Delta \vdash_{K'} B \Rightarrow \Delta \vdash_K B'$.
- (iv) Finally, if $B \in \Delta$, then B is already a WFF of K , and so (again) B cannot contain any of the new c_i , and once again $B' = B$, which ensures that $\Delta \vdash_{K'} B \Rightarrow \Delta \vdash_K B'$ in this case.

A parallel argument shows that $\Delta \vdash_{K'} \sim B \Rightarrow \Delta \vdash_K \sim B'$. So, if Γ is an inconsistent set of K' , then Γ is an inconsistent set of K . That is, if Γ is a consistent set of K , then Γ is also a consistent set of K' . \square

Lemma 2 for 45.17. Let A be a WFF of K in which v occurs free, and let c be a constant not occurring in A or in any proper axiom of K . Then, if $\vdash_K Ac/v$, then $\vdash_K A$.

Proof. We already proved this lemma (basically) in the course of proving 45.12. Here goes. Since $\vdash_K Ac/v$, we know there is a proof $\langle B_1, \dots, B_m \rangle$ of Ac/v in K . Let u be any variable that does not occur in this proof. As a result, note that u does not occur in A , unless $u = v$, since u does not occur in Ac/v . Let B'_i be the result of substituting u for c in B_i . That is, $B'_i = B_i u/c$. Now, $\langle B'_1, \dots, B'_m \rangle$ is also proof of in K . To see this, note that each B_i is either a logical axiom, a proper axiom, or an immediate consequence by modus ponens (MP) from two previous lines, and that this is also true for each B'_i . There are three cases:

- I. If B_i is a logical axiom, then so is B'_i . There are five sub-cases here [similar to (i.1)-(i.5), above]:
- (a) B_i is a logical axiom by K1-K3. Then, B_i is an instance of a propositional axiom schemata. In this case, $B'_i = B_i u/c$ is also an instance of a propositional axiom schemata, since merely substituting u for c in B_i cannot change the propositional logical form of any formula B_i .
- (b) B_i is a logical axiom by K4. So, B_i is of the form $\wedge vC \supset Ct/v$, where t is free for v in C . Thus, $B'_i = (\wedge vC \supset Ct/v)u/c = (\wedge vCu/c \supset (Ct/v)u/c) = (\wedge vCu/c \supset (Cu/c)t/v)$, where t is free for v in C . This is because u does not occur in B_i (hence, u does not occur in C , Ct/v , or t), and so $(Ct/v)u/c = (Cu/c)t/v$. Therefore, B'_i is of the form $\wedge vD \supset Dt/v$ (with $D = Cu/c$), where t is free for v in D . So, B'_i is also a logical axiom by K4.
- (c) B_i is a logical axiom by K5. So, B_i is of the form $C \supset \wedge vC$, where v does not occur free in C . Thus, $B'_i = (C \supset \wedge vC)u/c = (Cu/c \supset \wedge vCu/c)$. Therefore, B'_i is of the form $D \supset \wedge vD$ (with $D = Cu/c$), where v does not occur free in D . So, B'_i is also a logical axiom by K5.
- (d) B_i is a logical axiom by K6. So, B_i is of the form $\wedge v(C \supset D) \supset (\wedge vC \supset \wedge vD)$. Thus, $B'_i = (\wedge v(C \supset D) \supset (\wedge vC \supset \wedge vD))u/c = \wedge v(Cu/c \supset Du/c) \supset (\wedge vCu/c \supset \wedge vDu/c)$, which is also a logical axiom by K6.
- (e) B_i is a logical axiom by K7. So, B_i is of the form $\wedge vC$, where C is a logical axiom by K1-K6. Thus, $B'_i = \wedge vCu/c$, where Cu/c is a logical axiom by the above six arguments (which show that if C is an axiom by K1-K6, then so is $C' = Cu/c$). Hence, by K7, B'_i is also a logical axiom.

- II. If B_i is a proper axiom of K , then so is B'_i . For, if B_i is a proper axiom, then $B'_i = B_i u/c = B_i$, since c does not occur in any proper axiom of K (so “replacing” c with u in B_i does nothing to B_i).
- III. If B_i is an immediate consequence by MP of two previous lines in $\langle B_1, \dots, B_m \rangle$, then B'_i is an immediate consequence by MP of two previous lines in $\langle B'_1, \dots, B'_m \rangle$. Assume B_i is an immediate consequence by MP of two previous lines B_j and $B_j \supset B_i$. Then, $(B_j \supset B_i)' = (B'_j \supset B'_i)$, and B'_i will be an immediate consequence by MP of B'_j and $(B'_j \supset B'_i) = (B_j \supset B_i)'$.

1. $\vdash_K Ac/v$ [Assumption of Lemma 2]
2. $\vdash_K (Ac/v)u/c$ [(1), our proof above, and $(Ac/v)' = (Ac/v)u/c$]
3. $\vdash_K Au/v$ [(2), c does not occur in A]
4. $\vdash_K \wedge uAu/v$ [(3), metatheorem 45.4]
5. $\vdash_K \wedge uAu/v \supset (Au/v)v/u$ [Axiom K4, v is free for u in Au/v since either $u \notin A$ or $u = v$]
6. $\vdash_K \wedge uAu/v \supset A$ [(5), $(Au/v)v/u = A$]
7. $\vdash_K A$ [(4), (6), MP]

That completes the proof of Lemma 2. Now, we're ready to prove Gödel's metatheorem 45.17. □

45.17 (Gödel, 1930). Let Γ be a consistent set of WFFs of a first order theory K . Then, Γ is simultaneously satisfiable in a denumerable domain. That is, there is a denumerable interpretation I and a denumerable sequence s of elements of the domain of I such that s satisfies all members of Γ .

Proof. Let $\Gamma = \{A_1, A_2, \dots\}$ be a consistent set of WFFs of a first order theory K . Let $\langle v_1, v_2, \dots \rangle$ be an enumeration of the variables that occur free in Γ . Let K' be K plus denumerably many new constant symbols with an effective enumeration $\langle c_1, c_2, \dots \rangle$. And, let $\Gamma' = \Gamma c_i/v_i = \{A'_1, A'_2, \dots\} = \{A_1 c_i/v_i, A_2 c_i/v_i, \dots\}$. Now, we'll construct a denumerable interpretation I and a denumerable sequence s of elements of I 's domain such that s satisfies all members of Γ' and all members of Γ . The interpretation will be a denumerable *model* M of Γ' . That Γ' has a denumerable model is guaranteed by 45.16 (proved above), and the fact that:

- Γ' is a *consistent* set of *closed* WFFs of a first order theory K' .

Proof. It is obvious that Γ' is a set of *closed* formulas of K' . What we need to prove that Γ' is a *consistent* set of WFFs of K' . Assume, for *reductio*, that Γ' is an inconsistent set of WFFs of K' . Then, there exists a finite subset $\{A'_1, \dots, A'_n\} \subseteq \Gamma'$ such that $\{A'_1, \dots, A'_n\} \vdash_{K'} B$ and $\{A'_1, \dots, A'_n\} \vdash_{K'} \sim B$, for some WFF B of K' . So, applying the Deduction Theorem for K' n times (twice) yields:

$$(8) \quad \vdash_{K'} (A'_1 \supset (\dots (A'_n \supset B) \dots))$$

$$(9) \quad \vdash_{K'} (A'_1 \supset (\dots (A'_n \supset \sim B) \dots))$$

Let $X^* = Xv_i/c_i$, and recall $X' = Xc_i/v_i$. So, $X'^* = (Xc_i/v_i)v_i/c_i = X^{**} = (Xv_i/c_i)c_i/v_i = X$. Thus, we can re-write (8) and (9) in the following form, by re-writing B as $B^{*'}$:

$$(8) \quad \vdash_{K'} (A'_1 \supset (\dots (A'_n \supset B^{*'}) \dots))$$

$$(9) \quad \vdash_{K'} (A'_1 \supset (\dots (A'_n \supset \sim B^{*'}) \dots))$$

Now, letting $C = B^*$, we see that (8) and (9) are really of the following form:

$$(8) \quad \vdash_{K'} (A'_1 \supset (\dots (A'_n \supset C') \dots))$$

$$(9) \quad \vdash_{K'} (A'_1 \supset (\dots (A'_n \supset \sim C') \dots))$$

But, since $X' \supset Y' = (X \supset Y)' = (X \supset Y)c_i/v_i$, we can write (8), (9) in even more perspicuous form:

$$(8) \quad \vdash_{K'} (A_1 \supset (\dots (A_n \supset C) \dots))c_i/v_i$$

$$(9) \quad \vdash_{K'} (A_1 \supset (\dots (A_n \supset \sim C) \dots))c_i/v_i$$

Now, we can apply Lemma 2 to (8) and (9) in this perspicuous form, and infer:

$$(10) \quad \vdash_{K'} (A_1 \supset (\dots (A_n \supset C) \dots))$$

$$(11) \quad \vdash_{K'} (A_1 \supset (\dots (A_n \supset \sim C) \dots))$$

Then, applying the converse Deduction Theorem to (10) and (11) n times (twice) yields:

$$(12) \quad \{A_1, \dots, A_n\} \vdash_{K'} C$$

$$(13) \quad \{A_1, \dots, A_n\} \vdash_{K'} \sim C$$

So, $\{A_1, \dots, A_n\}$ is an inconsistent set of K' . And, since $\{A_1, \dots, A_n\} \subseteq \Gamma$, it follows that Γ is an inconsistent set of WFFs of K' . Hence, by the contrapositive of Lemma 1, we may infer that Γ is an inconsistent set of WFFs of K . Contradiction. So, Γ' is a *consistent* set of closed WFFs of K' . \square

Therefore, Γ' has a denumerable model M , by 45.16. Now, we construct a denumerable sequence $s = \langle s_1, \dots, s_m, \dots \rangle$ of elements of the domain D of M , as follows:

$s_m =$ the object assigned to c_i by M , if v_i is the m th variable in our enumeration of variables of K

If we run out of v_i 's and c_i 's occurring in the formulas of Γ and Γ' (or if there are gaps in the v_i 's and c_i 's occurring in the formulas of Γ and Γ' - see below for an example), then we will complete the construction of s (or fill-in the gaps in s) by repeating an arbitrary element d of the domain of M as often as necessary.

Since the elements of Γ' are all closed and M is a model of Γ' , s will satisfy all members of Γ' on M (*trivially*, since with closed formulas, we don't even need to look at the structure of s to realize that it will satisfy - on M - any closed formula that is true on M). And, *by design*, s will also satisfy all members of Γ on M . Why? Because, the members of Γ are just the members of Γ' , but with the c_i 's replaced by the v_i 's. And, this ensures that s will satisfy all members of Γ as well. To see this, think about the following example:

- $\Gamma = \{Fv_1, Gv_6v_7\}$. Then, $s = \langle c_1, d, d, d, d, c_6, c_7, d, d, \dots \rangle$, where d is some arbitrary member of the domain of interpretation M , which is a model of the set $\Gamma' = \{Fc_1, Gc_6c_7\}$. The sequence s satisfies all members of Γ . Why? Precisely because M is a model of Γ' . So, Fc_1 and Gc_6c_7 are both true on M . Hence, the object assigned to c_1 by M is in the extension of F , and the pair of objects assigned to $\langle c_6, c_7 \rangle$ by M is in the extension of G . So, the sequence s satisfies both $\{Fv_1, Gv_6v_7\}$ on M .

This s will always satisfy all members of Γ on M . Hence, any consistent set Γ of WFFs of K is simultaneously satisfiable in a denumerable domain (*i.e.*, each member of Γ will be satisfied by the sequence s , as defined above, in the domain of the model M of the set Γ' , whose existence is guaranteed by 45.16). \square

46.2. If $\Gamma \models_Q A$, then $\Gamma \vdash_{QS} A$. [Strong Completeness of QS]

Proof. Assume $\Gamma \models_Q A$. Then, by the definition of \models_Q , $\Gamma \cup \{\sim A\}$ is not simultaneously satisfiable. Thus, as a special case of Gödel's metatheorem (45.17) which we just proved above, $\Gamma \cup \{\sim A\}$ is not a consistent set of WFFs of QS. Therefore, by metatheorem (43.9), $\Gamma \vdash_{QS} A$. That completes the proof of the strong completeness of QS, which is the main metatheorem of quantifier logic. [As a corollary of strong completeness and strong soundness, we now have an exact correspondence between \models_Q and \vdash_{QS} , which was our aim.] \square