special constructions always stay within logicism? Whitehead was as pat as Russell in his lecture: 'The whole of mathematics is here', he announced confidently, 'it is mathematics, neither more nor less'.[66] But he gave no explicit description of this mathematics which the 'analytic stage' would deliver, so that the trivialisation of logicism is not definitively avoided.

I see a rather unfortunate line of influence from *Principia mathematica*, in that the philosophy of mathematics from that time to ours has become largely a cottage industry which in fact deals only with logic(s), set theory(ies), transfinite arithmetic and small pieces of other branches of mathematics. While extremely interesting material occurs within its own range, it comfortably avoids practically all mathematics that mathematicians do, and gives a highly distorted view of the *variety* of questions which can be raised in the philosophy of mathematics.[67] I am most perplexed by the reactions I receive from philosophers of mathematics to this criticism: the answer is either the pat and mathematically unproven 'it's all sets!', or the pat and philosophically uninteresting 'the point is philosophically uninteresting'. But with it left unanswered, "philosophy of mathematics" stays in its own little corner and so remains as mathematics for philosophers and philosophy for mathematicians—as we saw at the start of this paper.

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66 A.N. Whitehead, 'Presidential address; the organisation of thought', *Reports of the British Association for the Advancement of Science*, (1916: pb. 1917), 355–363; reprinted in *Proceedings of the Aristotelian Society*, n.s. 4 (1916–1917), 58–76; *The organisation of thought* (1917, London), ch. 6; and *The aims of education and other essays* (1929, London), ch. 8. My disclaimer of footnote 4 to attempt to discuss Whitehead's full views on mathematics are not infringed here, for in this lecture he dealt only with logicism.

67 Important antidotes have flown from the pen of G. Polyà (and others who follow him), especially concerning heuristics and mathematical proof, and also the undeservedly forgotten writings of F. Rostand on the vagueness of mathematical theories and the attempts to refine them.

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Skolem and the Löwenheim-Skolem Theorem: A Case Study of the Philosophical Significance of Mathematical Results

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The dream of a community of philosophers engaged in inquiry with shared standards of evidence and justification has long been with us. It has led some thinkers puzzled by our mathematical experience to look to mathematics for adjudication between competing views. I am skeptical of this approach and consider Skolem's philosophical uses of the Löwenheim-Skolem Theorem to exemplify it. I argue that these uses invariably beg the questions at issue. I say 'uses', because I claim further that Skolem shifted his position on the philosophical significance of the theorem as a result of a shift in his background beliefs. The nature of this shift and possible explanations for it are investigated. Ironically, Skolem's own case provides a historical example of the philosophical flexibility of his theorem.

> Our suspicion ought always to be aroused when a proof proves more than its means allow it. Something of this sort might be called 'a puffed-up proof'.
>
> Ludwig Wittgenstein, *Remarks on the foundations of mathematics* (revised edition), vol. 2, 21.

1. Introduction

If theories are not to go the way of science fiction, then they must be subject to certain constraints. I take this to be obvious as well as the view that, however little consensus there is on what the constraints of philosophical theories could be, there is recognition that philosophical reflection is also subject to this requirement. Unbridled philosophical fantasies, like feet on a frictionless floor, get nowhere.

Constraints, however, needed as they are, do not suffice to make philosophical reflection into the kind of inquiry many of its practitioners long for it to be. Our feet, though now secure on a frictionful floor, may take off in any number of opposing directions. What is needed, in addition, is a commonality of the constraints taken to be applicable to philosophical theories. Without a consensus on the considerations germane to a theory's confirmation, adjudication between competing views will prove futile. In fact, even talk of *competing* views in such cases becomes problematic. Philosophers must learn to walk along the same paths.

These requirements have been recognized only tacitly in the philosophy of mathematics. To date, there have been few responses to these demands. The view many thinkers have taken often is something akin to the one once voiced by an enthusiastic Bertrand Russell (*1901*, 75):

> In the whole philosophy of mathematics, which used to be at least as full of doubt as any other part of philosophy, order and certainty have replaced the confusion and hesitation which have formerly reigned. Philosophers, of course, have not yet discovered this fact, and continue to write on such subjects in the old way. But mathematicians, at least in Italy, have now the power of treating the principles of mathematics in an exact and masterly manner by means of which the certainty of mathematics extends also to mathematical philosophy.

On this view, some results in mathematics carry philosophical import of their own and can serve as fixed-points relative to which competing philosophies of mathematics can be judged. This idea was carried to the limit by some Hilbertians who looked forward to the day when, as Hilbert put it, 'Mathematics in a certain sense develops into a tribunal of arbitration, a supreme court that will decide questions of principle' (*1925*, 384). Von Neumann, for example, stressed 'the fact that this question [of the origins of the generally supposed absolute validity of classical mathematics], in and of itself philosophico-epistemological, is turning into a logico-mathematical one' (*1931*, 61).

The following discussion, focussing on the Löwenheim-Skolem Theorem (LST), is part of a critique of the view that mathematics provides a suitable source of constraints on philosophical reflections about it. It is the beginning of an examination of whether the LST yields support for a philosophical position, as many, Thoralf Skolem in particular, have claimed it does, or whether it only appears to because key aspects of the position have been tacitly assumed in the interpretation of the result. If the latter regularly turns out to be the case, then one might be tempted to conclude that the LST lacks independent philosophical significance and, taken alone, is irrelevant to most traditional philosophical disputes.

My inquiry will be limited to an examination of the very first attempt, Skolem's, to foist the LST into the philosophical fray. In the process, I will urge that a significant shift in Skolem's use of the LST took place. Aside from its general historical interest, this shift provides an actual, and therefore all the more striking, example of the LST's philosophical flexibility.

2. The Löwenheim-Skolem Theorem

The LST states that if there exists a model for a countable collection of sentences of some first-order language, that is, an interpretation that makes them true, then there exists an enumerable model for this collection, that is, a model whose universe of discourse contains at most denumerably many elements. A formal system any two models of which are isomorphic is called *categorical*. It follows from the LST that no formal system that has a non-denumerable model is categorical.

Skolem gave two proofs of this theorem. The first proof, published as *1920*, used the Axiom of Choice to *pare down* the universe of discourse of the original model to a countable number of elements. The second proof, published two years later (Skolem

1922) and closer in spirit to Löwenheim *1915*, does not make use of the Axiom of Choice. In fact, the second proof was offered to show the dispensability of this axiom for establishing the LST. Leaning crucially on the Axiom of Choice, the earlier result is stronger than this because the countable model it guarantees is a restriction of the original uncountable one. The second result provides no such guarantee and the countable model that is *built up* need bear no relationship to the original one. Really then, there are two distinct results and it is misleading to speak of *the* LST. Skolem, in particular, was at great pains to distinguish the two versions (see, e.g., *1922*, 293; and *1941*, 457–458) since he felt that inquiries into the foundations of set theory were best pursued agnostic with respect to its more controversial components, e.g. the Axiom of Choice. For this reason, Skolem confined his culling of foundational and philosophical consequences to the later result. In line with this, I will intend the 1922 result by 'LST'.

Skolem, notoriously, drew conclusions concerning the 'relativity' of set-theoretic notions. According to him, the relativity resides in the fact that a set may have a property in one model but lack it in another. For example, the set of all subsets of the natural numbers, the referent of '$\mathfrak{P}(\omega)$', is not enumerable according to the model guaranteed by the LST; this is so because a model renders every theorem of the theory true, and one of these states '$\mathfrak{P}(\omega)$ is not enumerable'. But since this model is countable, every set in its universe, including the referent of '$\mathfrak{P}(\omega)$', is countable as well.

This curious state of affairs is known as 'Skolem's paradox', despite Skolem's never having considered the situation paradoxical. Its 'resolution', presented by Skolem in his 1922 Address (*1922*, 295), consists in noting that to claim that a set s is countable is tacitly to make an existence claim of the form 'there exists a one-to-one function with domain the natural numbers and range s'. The referent of '$\mathfrak{P}(\omega)$' in the countable model is countable because there exists such a mapping between it and the natural numbers which, however, does *not* exist in the countable model. The countable model is blind to the fact that the set it has '$\mathfrak{P}(\omega)$' denote is countable in the intended model. This model's claim that $\mathfrak{P}(\omega)$ is uncountable, that is, that there exists no mapping of the appropriate kind (in the countable model), is true. In so far as claims of denumerability, equinumerosity, and finitude are tacitly existence claims, the corresponding properties, relations, as well as their negations are also said to be relative. Even the relation of equality may be said to be relative in this respect.[1]

Perhaps the following consideration will make the result seem even natural. The only subsets of a set that a model need 'see' are the ones that arise out of repeated (but at most countably many) applications of the set-construction operations permissible

1 Given the Axiom of Extensionality, we have

$$\alpha = \beta \longleftrightarrow (z)(z \varepsilon \alpha \longleftrightarrow z \varepsilon \beta).$$

However, in a non-transitive model (the existence of which is guaranteed) not all members of elements of the domain of the model are elements of the domain. Therefore, in such a model '$\alpha = \beta$' could be satisfied by $s1$, $s2$ whereas it might not be satisfied by this ordered pair in a transitive model if $s1$ and $s2$ do not share all their members.

in the theory. In a countable formalism, there will be at most a countable number of such operations. Therefore, this procedure can yield at most countably many sets. It is in this sense that the LST is an artifact of the countable nature of formal systems.

Before I begin, a few words of some historical interest aimed to prevent confusion when I turn to some of Skolem's remarks below. From our perspective, it is obvious that the possession of set-theoretic properties, if relative to anything, is relative to the model one is using to interpret the formal system. Several writers[2] have noted that this was perhaps not obvious to Skolem who, they claim, often interpreted the LST as demonstrating the relativity of set-theoretic notions to the axiom system itself (see, e.g., his *1929b*, 293; and *1958*, 635–637). This confusion, on first thought historically minor, gains in interest when one realizes that it went hand in hand with Skolem's inability to keep sharp the distinction between a syntactic formal system and its semantic interpretation, between 'is (un-)satisfiable' and 'is consistent' ('is contradictory'). Bernays (reported in Skolem *1970*, 22) has suggested that this was due to Skolem's training in the Boole/Schröder/Löwenheim/Korselt tradition of logic. This school did not consider logic to be a deductive system, with the consequences (i) that the difference between syntax and semantics was blurred and (ii) that the required sensitivity to the distinction between metalogic and logic—perhaps even the very idea of such a distinction—was lacking.

Undoubtedly, this state of affairs was responsible for Skolem's failure to prove the completeness theorem for quantificational logic (or even to consider the issue of completeness), even though he had the mathematical essentials for its proof eight years before Gödel presented his.[3] Viewed in this context, Skolem's (mis)formulations of the LST are not best looked upon as curiosities possessing whatever interest a great logician's confusions or errors may have. Rather, they are of significant historical interest in illuminating the conception of logic that Skolem had, perhaps one that was dominant in the early decades of this century.

Throughout his career, Skolem claimed that the LST had profound implications for the philosophy of mathematics.[4] It is often assumed that he always drew the same consequences from the LST. I will argue, however, that there was an important shift in his use of this result. A study of this shift will be quite suggestive in determining what the philosophical consequences of the LST might be.

2 This is suggested by W. Hart (*1970*, 107). The 'eminent logician' who also holds these views is presumably Hao Wang; see his 'A survey of Skolem's work in logic', in Skolem *1970*, 17–52 (p.40).

3 The reader can consult W. Goldfarb's *1979* for additional information.

4 Skolem, to repeat, was not alone in this belief. For example, von Neumann (*1925*, 412) wrote that

> The consequences of all this is that no categorical axiomatization of set theory seems to exist at all [. . .]. And since there is no axiom system for mathematics, geometry and so forth that does not presuppose set theory, there probably cannot be any categorically axiomatized infinite systems at all. This circumstance seems to me to be an argument for intuitionism.

For a more contemporary example, see Putnam *1980*.

4. Skolem's earlier views

During his early career, Skolem was an intuitionist of sorts. As will be shown shortly, he seemed to be committed to the view that, unless our faculty of mathematical understanding or intuition was capable of apprehending with clarity the existence of given objects or the validity of given inferential operations, these objects and operations could not be countenanced. He was an intuitionist in the sense that he believed that constraints in our capacity for creative mathematical reflection were constraints on which propositions could be asserted intelligibly and truthfully. Mathematical reflection did not merely discover facts but also determined which facts were around to discover.[5]

Unlike traditional intuitionists, however, Skolem accepted the idea of a foundation for classical mathematics. In his 1922 attack on the adequacy of axiomatic set theory (AST) for founding mathematics, he does not seem to question the need for one. Indeed, in 1919 he himself attempted to found arithmetic using the non-formal means of 'the recursive mode of thought' (*1923*, 304). Rather, he faulted AST and other formal systems because they failed to meet his criterion of adequacy for any foundation of mathematics. Roughly, if a system is to provide an adequate foundation for some domain, then (at least) it must be the case that all properties of the founding system can be considered properties of the founded domain.[6] The primary characteristics of the truths we are made aware of through the use of our faculty of intuition were clarity and absoluteness. Once arrived at, a truth was unequivocal. Consequently, the mental objects of mathematics, e.g. integers, and the rules of inference employed, e.g. mathematical induction, were 'immediately clear, natural, and not open to question' (*1922*, 299).

Skolem believed the LST guaranteed that AST failed the criterion of adequacy for foundational systems. AST, dealing as it does with relativized notions (such as 'is uncountable'), could not lay claim to the clarity, naturalness, and absoluteness required of a foundational system of arithmetic, and hence of all mathematics. For Skolem 'it was so clear that axiomatization in terms of sets was not a satisfactory ultimate foundation of mathematics' (*1922*, 300–301).

It is interesting to note that Skolem never considered the possibility of a foundation erected on the basis of some non-axiomatized notion of set and its properties, or the possibility of the development of some conception of set that would evade relativization and satisfy his criterion of adequacy. Why did he seem to rule out the possibility that our intuitive faculty of mathematical reflection might lead to some future theory of sets that would provide a foundation for mathematics, a foundation, the LST would then be taken to indicate, no axiomatized formal system could represent?

5 I mention, only to put aside, the interesting issue of whether Skolem's intuitionism was a product of Brouwer's influence or had some other source, perhaps the ideas of the school within which he was trained (see the text above). Since, however, Skolem remarked in (*1929a*, 217) that the ideas of his 1923 paper were developed 'independently of Brouwer and without knowing his writings' I am inclined to discount the first source.

6 Many qualifications are in order. For example, the properties in question should be non-epistemological, for clearly one wants to permit occasions when the founding system is epistemologically more perspicuous than the founded system or vice versa (e.g., what may be called 'Russell's trickle-down theory of psychological plausibility': see his *1913*). More could be said, but I think that this formulation is sufficient for the purposes to which I wish to put the criterion in this paper.

I conjecture that Skolem saw the set-theoretic paradoxes (e.g., Russell's) as proof positive that our intuitive mathematical faculty leads us astray about sets and that a reconstruction of naive set theory would have to begin, if it were to begin at all, by taking some linguistic structure as the touchstone of truth and existence. Skolem believed (*1950*, 524) that

> the set theoretic antinomies [...] scattered [*sic*—shattered?] the conviction that it was possible to find logical principles which were reliable. But, certainly, the mistake that the naive set theory was reliable does not prove that it should not be possible to detect the error in the classical set theoretic thinking and perhaps formulate a really correct reasoning,

relying essentially, according to Skolem, on some formal axiomatized system.[7]

This point reinforces my claim that the early Skolem was an intuitionist, albeit an idiosyncratic one. If mathematical intuition were viewed only as an instrument of discovery, there would have been no reason for him to infer the unintelligibility of a set's *really* being uncountable from the LST. That some truths may be undetectable by our mathematical telescope (and by the linguistic means available) would then be irrelevant to their status as truths. The relativization of all set-theoretic notions is a consequence of the LST *and* the view that, because our intuition leads to paradox when applied to these notions, formal axiomatized systems are the only handle we have on them.

In support of this analysis, I note the following sharp and revealing asymmetry. In *1922*, 295–296, Skolem remarks that the number sequence, defined as the intersection of all sets having the same inductive property, may be different in different models of ZF, and he always believed that 'this definition cannot [...] be conceived as having an absolute meaning, because the notion subset in the case of infinite sets can only be asserted to exist in a relative sense' (*1955*, 587). Yet, in contrast to his opposition to, or neglect of, set-theoretic foundations, such considerations did not prevent him from advancing a foundation for mathematics based on the integers, 'inductive inferences and recursive definition' (*1923*, 299–300). The reason is that Skolem did not believe we are forced to rely on formal linguistic characterizations of these notions; our faculty of intuition guides us along securely in our inquiry into their properties. The asymmetry, stems from his judgment, that 'the logical intuitions which gave rise to naive set theory are rather uncertain whereas the arithmetical

7 See, for example, (*1922*, 291) or (*1941*, 460) where he writes:

> La découverte des antinomies ayant montré clairement que la théorie simple des ensembles, due à Cantor, ne peut pas être maintenue, on a entrepris la restauration de la théorie des ensembles, soit par la voie axiomatique, soit au moyen de systèmes logico-formels. Les deux essais reviennent au fond au même.

According to Skolem, Cantor's naive set theory, the product of intuitive reflection on the notion 'set', sinned doubly by being vague and inconsistent (*1941*, 469), in contrast to arithmetic which, as noted, he considered 'clear, natural, and not open to question' (*1922*, 299).

ones known as recursive or inductive reasoning are quite clear and completely safe' (*1953*, 544).[8]

It is easy to see that Skolem and others might naturally have taken these reflections on the LST to reinforce a belief in our possession of a faculty of mathematical intuition whose creative but secure use was what enabled the doing of mathematics. If linguistic structures such as axiomatized formal systems could not supply the absoluteness that seemed so patent in many areas of mathematics, then its source must be sought elsewhere. What with the paucity of plausible or prominent candidates, this would naturally lead to a securing of intuition's position as guarantor of clarity, security and truth in mathematics. Indeed, this line of thought appears to have attracted von Neumann.[9]

Yet, a believer in the absoluteness of set-theoretic notions and the knowledge-independence of set-theoretic truth would find such use of the LST very suspect. Such an individual would not grant that the paradoxes of naive set theory gave our current linguistic structures the last word on which sets exist and which properties they may have. Perhaps our intuition can be 'corrected', or other as yet undreamt of means of inquiry may become available to us, or, finally, we just may never know what the facts about sets really are, these being accessible only to creatures with quite different constitutions. Skolem's move, from the paradoxes to making then current AST the arbitor of set-theoretic truth, *already* assumes the tacit rejection of this picture, a picture which he might have felt the LST undermined. On this reconstruction, it seems as if the LST's support of the naive intuitionist, over the naive realist, is purchased at the cost of tacitly assuming key elements of the former position while rejecting aspects of the latter.

However natural it would have been for Skolem to have believed the LST capable of adjudicating this dispute, it is difficult to be certain, so I will turn to an issue on which the early Skolem certainly felt the LST bore, namely, whether AST is an appropriate foundation for mathematics. On this front, his opponent is one who believes that all thoughts, mathematical or other, must be expressible in language. Although it may be very difficult to place one's thoughts into words, it is incoherent to

8 Skolem's basic position on these issues has been subject to misinterpretations. E.g., Resnik states that 'Skolem's own conclusion [...was] that the standard axiomatic set theories contain sets which are uncountable only relative to these set theories *but which are countable from an absolute point of view*' (*1965*, 425, italics inserted). The italicized clause is simply false. Skolem always writes of the relativity of all set-theoretic notions (see, for instance, some of the quotations reprinted below). It would be antithetical to his whole approach to set theory (as opposed, for example, to arithmetic) to suppose the intelligibility of 'an absolute point of view' from which one may determine what the properties of a given set really are. (If Skolem does somewhere write of 'absolute countability', then, instead of taking him to be making reference to the set-theoretic notion of countability, one would have to interpret him as referring to some notion of countability given to one through reflection on the construction of the number sequence by one's faculty of intuition. The issue may not arise since Resnik cites no references, and I know of none.)

9 See footnote 4. That this is in fact what happened is very hard to substantiate directly. It is, however, completely consistent with the available evidence. Skolem, according to his own report, proved the LST in 1915–1916 (*1922*, 300–301), though he only published it in 1920; the beginnings of his attempt to found mathematics on the basis of his non-formal 'recursive mode of thought' were completed by 1919 (though only published in 1923; see *1923*, 332).

countenance thoughts which must forever elude such linguistic cloaking. Such an opponent deems it illusory to imagine that we possess a magical faculty of mathematical intuition that permits us to apprehend thoughts that cannot be carried by any linguistic vehicle. In mathematics, such a characterization would be in terms of some formal axiomatized system like AST. This opponent would insist that if the notion of foundation is to play any role at all, then it cannot consist of truths that must resist characterization by all linguistic means, in particular by all formal axiomatizations of our knowledge of the domain in question. Such an opponent need not question the relativity of all set-theoretic notions; nor need she reject Skolem's criterion of foundational adequacy. Rather, she would point to the fact that Skolem enters the argument with a particular conception of the nature of mathematical truth, namely, that, in some cases, we are made aware of it through the clear and secure operation of a faculty of intuition which permits us to apprehend objects and thoughts that resist complete linguistic characterization. It seems as if this background position is needed to get the LST to render Skolem's verdict on the case of AST's foundational adequacy. But this assumption would be hotly disputed and, indeed, is part of what is at issue. In short, if someone were convinced by his argument, then it would be dispensable since he already must have assumed something like its conclusion in order to make it cogent.

Again, it is not clear that the LST can be brought to decide between these competing positions without begging the question. Skolem, at least, was unsuccessful in doing this. It is so ironic that a prime example of a historical figure who adopted this anti-early-Skolem position is Skolem himself. He shifted his use of the LST from a weapon against the foundational adequacy of AST to one wielded against the intelligibility of the language-independence of mathematical truth. I will now turn to an articulation and a defense of this claim.

4. Skolem's later position

Over the years, Skolem abandoned his early conclusions about the adequacy of formal systems for foundational purposes. Whereas before, he believed that AST was not a suitable basis for mathematics, later (*1958*, 635), he could

> not understand why most mathematicians and logicians do not seem satisfied with this idea of sets defined by a formal system, but, on the contrary, speak of the insufficiency of the axiomatic method. Naturally, this idea of set has a relative nature since it depends on the chosen formal system. But if this system is suitably chosen, one can nevertheless develop mathematics taking it as a basis.

Skolem not only left open the possibility that formal systems, like AST, are of some use in foundational research, but went further and urged that they *should* be employed in any serious analysis of 'mathematical thought'. He declared that his 'point of view is [. . .] that one *should* use formal systems for the development of mathematical ideas' (*1958*, 634, italics added; see also p.637). In fact, his shift on the foundational adequacy of formal systems was so great that he could claim that 'one of the most important achievements in modern foundational research is the perfection of the axiomatic method known as the notion of formal system or logical language' (*1953*, 545–546) and declare that 'his conception [of 'the fundamental mathematical notions'] is founded above all on the idea of *systems* or *formal language*' (*1958*, 633).[10]

Why did Skolem repudiate his earlier conclusions on this issue? It is tempting to think that his work on, and ultimately his proof of the existence of, non-standard models of arithmetic in *1933* and *1934* shook his faith in the intuitive grasp he felt we had on the numbers and their properties, thus leading him to lay more foundational emphasis on formal systems.

This explanation fails on several counts. For one thing, it is false to the historical facts since Skolem, early and late, held that our faculty of intuition was at home with the integers. (For supporting passages and a discussion of how consistent this position is with some of his other later views, see below.) For another thing, this explanation simply begs the relevant question since he never would have felt our intuitive grasp on the numbers threatened by the existence of non-standard models *unless* he already felt our understanding was given us by some axiomatic system, in this case the Dedekind-Peano axioms. Recall how unperturbed Skolem was in *1922* when noting that different models of ZF may take different sets to be ω.

One highly relevant factor was the increasing importance formal systems took on in foundational studies and mathematical logic. The intensive work on Hilbert's program by many young and gifted mathematicians in the late 1920s is surely significant. Out of this research came not only particular results, but also a greater understanding of what a fully formalized and axiomatized system is. When these investigations culminated with Gödel's epochal paper *1931*, it would have been quite difficult for a researcher in the foundations of mathematics to deny the importance of formal systems. It really was Gödel's work (his *1931* and his work in *1930* on the completeness of quantificational logic) that signalled the general understanding of the modern distinctions between syntax and semantics, and between theory and meta-theory, and, hence, of a formal system.

The waxing of this conception saw the waning of the older tradition of logic, the one in which Skolem began his career. The reversals in his estimation of the foundational value of formal systems in general, and AST in particular, illustrate well the extent to which he was a transitional figure in the changing conception of logic that took place in the 1920s and 1930s.[11]

In sum, Skolem now believed that there were no linguistically disembodied mathematical thoughts. Although he continued to hold that mathematical objects were the creations of human minds, he no longer claimed that this activity and its results were undescribable by linguistic means. In (*1958*, 636), he wrote that 'It is a misunderstanding to speak of the insufficiency of the axiomatic method. Because mathematical objects are nothing but human thoughts and therefore the existence of

10 Failure to note this shift seems universal. It is, for example, a drawback of Hart *1970*. Goldfarb, also, repeatedly, but incorrectly, claims that 'Skolem was a constant opponent of all formalist and logicist foundational programs' (*1979*, 358; see also p.364).

11 The reader is advised to consult Goldfarb *1979*.

these objects naturally is limited as are the possible logical operations'. He now deemed axiomatized theories 'sufficient' to express all truths about mathematical objects and 'the possible logical operations' that could act on these truths to generate yet further truths. Truths and 'operations' not so formulable were considered 'impossible' (*1941*, 470).

Putting aside explanations for this about-face, what implications did it have for Skolem? Firstly, one should expect it to have consequences for the kind of mathematical work that he would consider undertaking. Indeed, in 1950, Skolem, who in 1922 criticized those who thought AST provided a foundation for arithmetic, reports that he had 'many years ago made an attempt to base arithmetic on RTT [ramified type theory] but did not succeed very well at that time. Recently I tried again with better results' (*1950*, 527).

Secondly, and more relevantly, one should expect this change of background to alter the philosophical consequences drawn from the LST. It would seem that if formal systems like AST provide suitable foundations for mathematics, perhaps even necessary ones, then, on the assumption that the LST entails the relativity of set-theoretic notions, we should abandon a language-independent view of mathematical truth. In fact, this is exactly the position that Skolem took. In (*1941*, 468), he wrote that 'The true significance of Löwenheim's theorem [the LST] is precisely this critique of the undemonstrable absolute'. It makes as much sense to ask whether a mathematical entity *really* has some property or not (e.g., whether a set *really* is finite or not) as it does to ask whether the temperature of a liquid *really* is 0° or not. Relatedly, for Skolem, there is no such thing as *the* set of all real numbers *tout court*. Relative to a choice of a formal system and a choice of a model, we can speak of the referent of a syntactic expression, in this case '$\mathfrak{R}$', of the theory; until then, 'one does not know what the author really means' (*1953*, 583).[12]

In general, Skolem later viewed the LST as dealing a death-blow to the plausibility that there are *any* absolute (in particular, language-transcendent) mathematical truths to be captured or given a foundation. On the contrary, 'A consequence of this state of affairs [the LST] is the impossibility of absolute categoricity of *the fundamental mathematical notions*' (*1958*, 635, italics added). According to him, it followed that 'All the notions of set theory, *and consequently of all of mathematics*, find themselves in this way *relativized*. The meaning of these notions is not absolute; it is relativized to the axiomatic model' (*1941*, 468; italics added). He argued that 'if one analyzes mathematical reasoning in such a way as to formulate the fundamental modes of thought as axioms'—something the later Skolem urged (see, e.g., *1941*, 470; *1958*, 634, 635, 636—all quoted in the text)—'then the relativism is inevitable because of the general nature of Löwenheim's theorem' (*1941*, 468). He concluded

12 Resnik *1965* saddles the 'Skolemite' with the view that the set of real numbers is absolutely countable. As far as Skolem, the arch 'Skolemite', is concerned, this is doubly misguided. On his later view, the referents of many expressions in a theory, as well as *all* set-theoretic notions, are relative to choices of formal system and interpretation. Pending these, Skolem just 'does not know what the author really means' (*1955*, 583) by such words as 'is countable' or 'the set of real numbers' (see footnote 8). Independently of such choices, there is no right answer to the questions what the correct notion of countability is or what the real set of real numbers is.

that a relativist conception of the fundamental mathematical notions 'is clearer than the absolutist and platonist conception that dominates classical mathematics' (*1958*, 633). In (*1941*, 470), in an apparent reference to his earlier views, he wrote:

> That axiomatization leads to relativism is sometimes considered to be the weak point of the axiomatic method. But without any reason. Analyzing mathematical thought, and fixing the fundamental hypotheses and modes of reasoning could not but be advantageous to the science. *It is not* a weakness of a scientific method that it cannot give us the impossible.

That this is the best reconstruction of Skolem's later views is obscured by the fact that the historical Skolem lapses into incoherence. I will take a moment to explain why I think this is so, and why I believe that the above analysis resolves the incoherence in the most satisfactory manner.

The problem arises because Skolem, on a few occasions, falls back into talk of our clear and secure intuition which provides us with *the* foundation for arithmetic. In (*1950*, 527), discussing his disappointment that so few logicians had attempted to develop as a foundation for mathematics the system of primitive recursive arithmetic that he laid out in his *1923*, he wrote: 'When I wrote my article I hoped that the very natural feature of my considerations would convince people that this finitistic treatment of mathematics was not only a possible one but *the* true or correct one—at least for arithmetic'. It is true, he continued, that, adopting such a foundation, much of present-day mathematics would then be lost. 'The question is, however, what we shall lose or gain by such a change. As to clearness and security we certainly only gain much' (*1950*, 527). This, together with his other views, generates an inconsistency. One cannot hold simultaneously (1) that AST can and should provide a foundation for all of mathematics (including arithmetic); (2) that the LST guarantees the relativity of all set-theoretic notions; (3) that the truths of the founding system are of a kind with the truths of the founded domain (i.e., the criterion of foundational adequacy); and (4) that the truths of primitive recursive arithmetic can be seen to be clearer and more secure than any others (in particular, than those of AST). The interpretive principle that enjoins one to do minimum damage to Skolem's views requires that we restrict our discussion of what to reject to (3) and (4). I shall argue that, though either choice involves problems, we should reject (4).

The costs involved in reinterpreting Skolem as rejecting (4) are rather straightforward; we must explain away the odd occasions when Skolem writes as quoted above. This task is facilitated slightly once one realizes that the above quotation mildly misleads in several respects. First, it is not clear whether Skolem merely was reporting what his intentions were in *1923* or whether, in addition, he was endorsing and reaffirming them. In support of the first interpretation, we find Skolem assuring us later that his is the spirit of tolerance. 'I am no fanatic', he wrote (*1950*, 527), 'and it is not my intention to condemn the nonfinitistic ideas and methods'. His point then was not so much one of 'good or bad' but of 'better or worse'. Secondly, when Skolem wrote of 'finitistic mathematics', although it seems that he was referring to his non-formal 'recursive mode of thought' of *1923*, he really had *formalizations* of primitive recursive arithmetic in mind. He approvingly cited Curry's paper on 'A formalization of recursive arithmetic' (*1950*, 526) and himself referred to recursive arithmetic as a formal language (*1958*, 634). Skolem's assertion now reduces (at worst) to the claim that, though many formal systems may do the foundational trick, a foundation based on a formalization of primitive recursive arithmetic would be clearer and more secure, this clarity and security perhaps being guaranteed by some faculty of mathematical intuition. Though this claim still does not sit very well with (1)–(3), the costs involved in rejecting it are not intolerably high; that is, Skolem's position remains an interesting one and some account can be given of why occasionally he seemed to favor primitive recursive arithmetic as a foundation over other (now formal) systems. We can, at least in part, attribute this partiality to the fact that he was the originator of this arithmetic and that it had received, at least in his opinion, insufficient attention (*1950*, 526–527).

The situation is quite different when we consider rejecting (3). In the first place, doing so leaves one without any account of why AST's being a good (even a necessary) foundation for mathematics leads to the relativity of 'fundamental mathematical notions' (*1958*, 635). The criterion of adequacy assured us that if F provided an adequate foundation for some domain D and the truths of F have property P, then the truths of D have property P. We have seen that Skolem, early and late, assumed some version of this.

More deeply, the criterion of adequacy seems to be built into the historical notion of a foundation in that, usually, F is offered as a foundation for D when the nature of the truths of D with respect to some property P is not known but there is confidence about whether the truths of F possess P or not. Thus, Frege believed that a reduction of arithmetic to logic would decide the issue whether the truths of arithmetic were analytic or not since the truths of logic clearly were. And Hilbert attempted to found classical mathematics on finitary mathematics (by using the latter to give a consistency proof of the former) with the expectation of showing that all finitary truths derived classically could be proved by finitary means alone. If one rejects the criterion of adequacy, then one prominent historical motivation for foundations is lost, since the nature of the truths of the founding system may differ utterly from those of the founded system.

In short, the costs of securing the consistency of Skolem's position by jettisoning (3) are very great. Doing so would leave Skolem with a rather uninteresting position philosophically, since a historically important philosophical rationale for seeking mathematical foundations would be undercut, a rationale which he always accepted tacitly. For these reasons, the above reconstruction of his views best preserves their philosophical interest and does least damage to the historical record (though, of course, any reconstruction that renders Skolem consistent will have to do some such damage).

What is of particular interest in this context is the shift in Skolem's background assumptions and the concomitant shift in his philosophical employment of the LST. Whereas the early Skolem began by assuming the language-transcendence of certain mathematical truths, e.g. those of arithmetic, and argued that the LST rendered doubtful the foundational utility of AST, the later Skolem affirmed the foundational indispensability of formalized, axiomatized theories and used the LST to urge that all mathematical truth is relative and language-dependent. Naturally, the later Skolem's argument has no more force against one who starts with a language-transcendent view of mathematical truth, than his earlier arguments have against a committed 'relativist'. The tacit assumption that mathematics can do with (and, indeed, cannot do without) a formalized, axiomatized foundation, where it is understood that an acceptable foundation must meet the criterion of foundational adequacy, would be rejected by the believer in the language-independence of mathematical truth; it seems as if the premises needed to draw the conclusion of relativism from the LST would be as unpalatable to the absolutist as the conclusion itself.

This historical case study exhibits the philosophical leeway of the LST and suggests that its philosophical consequences are parasitical on the philosophical views one conjoins with it. In the writings of both the early and the late Skolem, there are instances of philosophical positions deriving support allegedly from a mathematical result that seems philosophically lifeless without the infusion of precisely those or related positions.

5. Conclusion

Obviously, not all the philosophical possibilities of the LST have been explored, only those Skolem thought it had.[13] Though further exploration is in order, one should expect that other attempts to distill potent philosophical spirits directly from the bare LST (or related results) will fare no better than Skolem's. The more one examines the case of the LST, the more skeptical one becomes of the existence of precious philosophical nodules lying beneath the surface just waiting to be dug up.

As mentioned earlier, some thinkers, following Russell's lead, have hoped that mathematics could provide constraints for the philosophy of mathematics; that is, they have hoped that if philosophers just could formulate their questions precisely enough and attend to the doings of mathematicians carefully enough, then answers would be forthcoming finally to their problems. Yet, one must recognize that mathematics is just another aspect of the total human experience that so perplexes and leads to philosophy. Mathematical experience is no more philosophically self-interpreting than any other. For the most part, we do not understand the full philosophical significance of a mathematical truth (or any other truth for that matter) until we understand the role it plays in an articulated theory that brings together in harmonious fashion various central conceptions.

A mathematical result shines only when illuminated by other views. When these are simple as they are in Skolem's use of the LST, the shine is more like a crude reflection, as sections 3 and 4 suggest. It remains to be seen whether aspects of our mathematical experience, including, perhaps, particular of its products, can be brought together in a sophisticated, satisfying, and self-reinforcing manner.

13 For example, Putnam's recent use *1980* of this theorem has not been addressed.

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This essay is dedicated to the cherished memory of Nelson.

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