

## *k*-Validity vs Validity in *Q*: A formula of *Q* that is *k*-valid for all *k*, but not valid

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Consider the following three formulas of *Q* [where, as always,  $p \wedge q \stackrel{\text{def}}{=} \sim(p \supset \sim q)$ , and  $\bigvee p \stackrel{\text{def}}{=} \sim \bigwedge \sim p$ ]:

$$p \stackrel{\text{def}}{=} \bigwedge x' \bigwedge x'' \bigwedge x''' [(F^{***} x' x'' \wedge F^{***} x'' x''') \supset F^{***} x' x''']$$

- In more standard notation:  $p \stackrel{\text{def}}{=} (\forall x)(\forall y)(\forall z)[(Rxy \wedge Ryz) \supset Rxz]$ .

$$q \stackrel{\text{def}}{=} \bigwedge x' \bigvee x'' F^{***} x' x''$$

- In more standard notation:  $q \stackrel{\text{def}}{=} (\forall x)(\exists y)Rxy$ .

$$r \stackrel{\text{def}}{=} \bigvee x' F^{***} x' x'$$

- In more standard notation:  $r \stackrel{\text{def}}{=} (\exists x)Rxx$ .

Informally, *p* asserts that the 2-place relation  $F^{***}$  (*R*) is *transitive*, *q* asserts that  $F^{***}$  (*R*) is *serial*, and *r* asserts that there is some object that bears the relation  $F^{***}$  (*R*) to itself. Now, consider the following complex statement, constructed out of *p*, *q*, and *r*:

$$A \stackrel{\text{def}}{=} (p \wedge q) \supset r$$

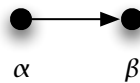
Claim *A* asserts that if  $F^{***}$  (*R*) is transitive and serial, then some object bears  $F^{***}$  (*R*) to itself.

**Fact.** *A* is *k*-valid for all (finite) *k*, but *A* is not valid [ $\neq_Q A$ ].

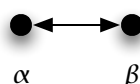
*Proof.* First, we will show (informally) that *A* is *k*-valid, for all *k*. Our (informal) argument will involve showing that *A* is true on all 1-element interpretations, and all 2-element interpretations, and ..., and all *k*-element interpretations, for all *k*. We will do this by arguing (informally) that we cannot make *A* false on any *k*-element interpretation. And, since *A* is a closed formula, it must either be true or false on each interpretation of *Q*. Thus, it will follow that *A* is true on all *k*-element interpretations of *Q*, for all *k*.

Let's think about 1-element interpretations  $I_1$  first. In order to make *A* false on any interpretation *I*, we would need to make *p* and *q* both true, and *r* false on *I* (these are just the falsity-conditions for  $\supset$ ). And, to make *r* false on a 1-element interpretation  $I_1$ , we need to ensure that its single element  $\alpha$  is such that  $\sim R\alpha\alpha$ . But, we also need to ensure that *q* is true on  $I_1$ . Thus, we need there to be some element  $\beta$  of  $I_1$  such that  $R\alpha\beta$  is true on  $I_1$ . Because there is only one element in the interpretation, we cannot make *q* true while *r* is false. This shows (already, and without even considering that *p* must also be true on  $I_1$  in order to make *A* false on  $I_1$ ) that there can be no 1-element interpretation  $I_1$  on which *A* is false. In other words, we have just shown that the formula *A* is 1-valid (indeed, we've even shown that  $q \supset r$  is 1-valid).

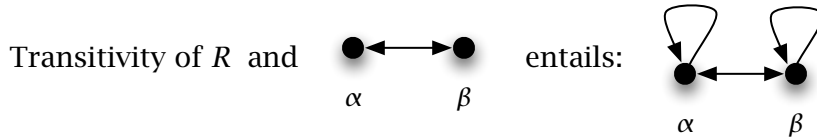
Next, we need to argue that we cannot make *A* false on a 2-element interpretation  $I_2$  either. So far, we have two elements  $\alpha$  and  $\beta$ , with the following structure (arrows represent *R*-relations):



Now, because we need *r* to remain false on  $I_2$ , we must have  $\sim R\beta\beta$  on  $I_2$ . And, because we need *q* to remain true on  $I_2$ , we need there to be some  $\gamma$  such that  $R\beta\gamma$  is true on  $I_2$ . The only way to do this (without adding yet another element to our interpretation  $I_2$ ) is to try to make  $R\beta\alpha$  true on  $I_2$ , which would yield the following (symmetric) structure:

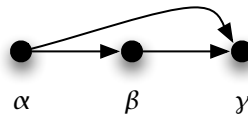


This structure  $I_2$  is one on which  $q$  is true and  $r$  is false. But, now we have a problem with ensuring that  $p$  is true on  $I_2$ . If the truth of  $p$  were enforced here, then we would end-up with both  $R\alpha\alpha$  and  $R\beta\beta$ . That is:



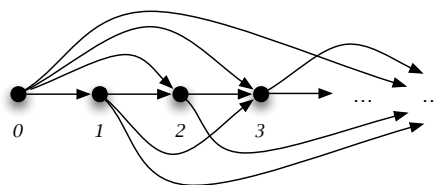
To see this, just look at the instance of  $p$ , where  $x' = \alpha$ ,  $x'' = \beta$ , and  $x''' = \alpha$  (and then the instance of  $p$ , where  $x' = \beta$ ,  $x'' = \alpha$ , and  $x''' = \beta$ ). Therefore, it is impossible to make  $A$  false on an interpretation containing only two elements  $I_2$ . That is, we have just shown that  $A$  is 2-valid.

Perhaps we can make  $A$  false on an interpretation with *three* elements  $I_3$ ? We have just seen that in order to make  $p$  true, while  $q$  is true and  $r$  is false, we need to add a *third* object  $\gamma$  to  $I_2$ , which would yield an interpretation  $I_3$  with the following relational structure:



Again, in order to make  $r$  false on  $I_3$ , we must have  $\sim R\gamma\gamma$  on  $I_3$ , and in order to make  $q$  true on  $I_3$ , there must be *some* object  $\delta$  such that  $R\gamma\delta$ . We could *try* to satisfy this constraint by forcing either  $R\gamma\alpha$  or  $R\gamma\beta$  on  $I_3$ . But, both of these choices will end-up with the same sort of inconsistency with  $p$  and  $\sim r$  that we just saw in the previous ( $k = 2$ ) case, with the introduction of object  $\beta$ . That is, if we enforce  $R\gamma\alpha$ , then  $p$  will entail  $R\alpha\alpha$  and  $R\gamma\gamma$  (hence  $r$  will be true), and if we enforce  $R\gamma\beta$ , then  $p$  will entail  $R\beta\beta$  and  $R\gamma\gamma$  (and  $r$  will be true). And, this frustrating story will repeat itself, no matter how large the (finite) domain of  $I_k$  is allowed to be —  $A$  cannot be false on any interpretation of (finite) size  $k$ .  $\therefore A$  is  $k$ -valid, for all  $k$ .

The second thing we need to demonstrate is that  $A$  is *not* valid [i.e.,  $\neq_Q A$ ]. Remember, just because all *finite* interpretations have a certain property, it doesn't follow that all *infinite* interpretations must also have that property (just think about the property of *being finite*!). So, just because  $A$  is true on all finite interpretations (as the informal argument above shows), it doesn't follow that  $A$  is true on all infinite interpretations as well. And, in fact,  $A$  is false on some infinite interpretations. Let  $I_\infty$  be an interpretation of  $Q$  in which  $D = \mathbb{N}$ , and  $F^{**'}$  gets assigned the relation  $Rx\gamma \stackrel{\text{def}}{=} x < \gamma$  by  $I_\infty$ . Here's a "picture" of  $I_\infty$ :



$A$  is false on  $I_\infty$ . To demonstrate this, we need to show that  $p$  and  $q$  are both true on  $I_\infty$ , but  $r$  is false on  $I_\infty$ . It is easy to see that both  $p$  and  $q$  are true on  $I_\infty$ , since the less-than relation is clearly both transitive and serial on the natural numbers. And, it is also clear that  $r$  is false on  $I_\infty$ , since *no* natural number is less than itself. This completes the (informal) proof that  $A$  is  $k$ -valid, for all (finite)  $k$ , but  $A$  is invalid.  $\square$

**Addendum:** Can you give more *rigorous* proofs of these two claims about  $A$ ? Presumably, the first claim would proceed *via* induction on the size  $k$  of candidate interpretations  $I_k$ . We have already established the basis ( $k = 1$ ) case, above. The inductive hypothesis would be that  $A$  is false on all interpretations  $I_j$  of size  $1 \leq j < k$ . And, from this assumption, the goal would be to prove that  $A$  is false on interpretations  $I_k$  of size equal to  $k$ . You can see how that argument is going to run, just by thinking about how we were led into frustrating inconsistencies above, when we tried to make  $A$  true by adding one element to an interpretation  $I_j$  on which  $A$  was false. For the second claim, can this one be proved (in an illuminating way) by induction? Or does it involve properties so basic to the less-than relation (over natural numbers) that it would be difficult to see how an illuminating "inductive proof" would go?