Philosophy 140A Take-Home Mid-Term
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You are to answer all six (6) exercises on this take-home exam. Your solutions are **due on Tuesday, April 10 at 4pm**. You may work in groups on this exam (with the usual rules and procedures for group work).

1 Formalizing Some of the Metatheory of P in Q

1.1 The Formal System PS' for P

Consider the following formal system for P, which I will call PS'. The system PS' has the same three axiom schemata (PS1)–(PS3) that Hunter's formal system PS has, and it has the following single rule of inference:

(MP′) From $\vdash_{PS'} A$ and $\vdash_{PS'} A \supset B$, infer $\vdash_{PS'} B$.

So, the only difference between PS and PS' is that the (MP) rule of PS does *not* require its premises to be theorems of PS, whereas the (MP′) rule of PS' *does* require its premises to be theorems of PS'.

Exercise #1. Explain why PS and PS' have exactly the same set of theorems.
Exercise #2. Explain why PS and PS' are (nonetheless) *not* the same formal system.

1.2 Formalizing Some of the Metatheory of PS' in Q

Consider the following four universally quantified WFFs of Q:

(1) $\bigwedge x' \bigwedge x'' F^{*\prime} f^{**\prime} x' f^{**\prime} x'' x'$

(2) $\bigwedge x' \bigwedge x'' \bigwedge x''' F^{*\prime} f^{**\prime} f^{**\prime} x' f^{**\prime} x'' x''' f^{**\prime} f^{**\prime} x' x'' f^{**\prime} x' x'''$

(3) $\bigwedge x' \bigwedge x'' F^{*\prime} f^{**\prime} f^{**\prime} f^{*\prime} x' f^{*\prime} x'' f^{**\prime} x'' x'$

(4) $\bigwedge x' \bigwedge x'' (F^{*\prime} x' \supset (F^{*\prime} f^{**\prime} x' x'' \supset F^{*\prime} x''))$

Now, consider the following interpretation I of Q.[1]

The domain D of I is the set of WFFs of P.
"$F^{*\prime}$" gets interpreted by I as the (metatheoretic) property "is a theorem of PS'" (*i.e.*, $\vdash_{PS'}$).
"$f^{*\prime}$" gets interpreted by I as the "$\sim$" connective in PS'.
"$f^{**\prime}$" gets interpreted by I as the "$\supset$" connective in PS'.

Exercise #3. Explain why (1)–(4) are all *true* on I. [**Hint**: you might want to do Exercise #4 first.]
Exercise #4. Describe a procedure for translating schematic metatheoretic statements of the form "$\vdash_{PS'} S$" ("S is a theorem *schemata* of PS'") into universally quantified WFFs of Q (assuming the I-interpretations of $f^{*\prime}$, $f^{**\prime}$, and $F^{*\prime}$). And, explain why the Q-translation of any *true* metatheoretic statement of this form must be *true on* I. [Example: the Q-translation of "$\vdash_{PS'} A \supset A$" should come out as "$\bigwedge x' F^{*\prime} f^{**\prime} x' x'$".]

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[1] Strictly speaking, we should also say what I assigns to (*i*) the constant symbols of Q, (*ii*) the propositional symbols of Q, and (*iii*) the other predicate and function symbols of Q. But, since these aspects of I will not matter for the question at hand, I have not bothered to specify them. You could, for instance, let a_i denote the ith formula in some enumeration of P's WFFs. And, you could assign any properties/functions you like to all the other predicate/function symbols in Q. Moreover, you can let I assign whatever truth-values you want to Q's propositional symbols. Be sure not to confuse the propositional symbols of P —- which are in the domain of I — with the propositional symbols of Q, which are *not*. Also, do not confuse the connectives of P with the connectives of Q. The connectives of Q are — on I — being used to express connectives in the *metalanguage* of P! It is very important to stay clear on object–language *vs* meta–language in this problem!

Now, consider the following, different interpretation I' of Q:

The domain D of I' is the following set of three natural numbers: $\{0, 1, 2\}$.

"$F^{*\prime}$" gets interpreted by I' as the property "is identical to the number zero".

"$f^{*\prime}$" gets interpreted by I' as the 1-place function f_1 with the following matrix:

x	0	1	2
$f_1(x)$	1	1	0

"$f^{**\prime}$" gets interpreted by I' as the 2-place function f_2 with the following matrix:

f_2	0	1	2
0	0	2	2
1	2	2	0
2	0	0	0

Exercise #5. Show that (2)–(4) are all *true* on I', but (1) is *false* on I'. And, explain how this could be used to show that (PS1) is *independent* of {(PS2), (PS3), (MP′)}. [**Hints**: Hunter's discussion on pages 123–124 and my handout on Hiż should both be useful for #5. What you'll need to do here is show that the Q-translations of all theorem schemata of the system {(PS2), (PS3), (MP′)} are true on I', but that the Q-translation of (PS1) is false on I'. Hunter does the hard part of this on pages 123-124 (the easy part is writing down the Q-translations). This will imply the existence of a property (the truth-on-I' of their Q-translation) that all theorem schemata of the system {(PS2), (PS3), (MP′)} have, but that (PS1) lacks. This is sufficient to show that (PS1) is not a theorem of the system {(PS2), (PS3), (MP′)}.]

2 Completeness of Another Formal System for Propositional Logic

Consider a language P^* that is similar to P but has as its two connectives $\sim$ and & (negation and conjunction) rather than $\sim$ and $\supset$. Of course, the axioms and rules of inference of the formal system for P^* (call it PS^*) are going to be different from the ones for our (PS), since the axioms and rules of (PS) can't even be expressed in P^*. Assume that the following five (5) *schemas* are rules of inference in the system (PS^*):

(PS^*1) $\{A, \sim A\} \vdash_{PS^*} B$ [*i.e.*, from A and $\sim A$, infer B, for any WFFs A and B of P^*]

(PS^*2) $\{\sim(\sim A \,\&\, B), \sim(\sim A \,\&\, \sim B)\} \vdash_{PS^*} A$ [*i.e.*, from $\sim(\sim A \,\&\, B)$ and $\sim(\sim A \,\&\, \sim B)$, infer A]

(PS^*3) $A \,\&\, B \vdash_{PS^*} A$ [*i.e.*, from $A \,\&\, B$, infer A]

(PS^*4) $A \,\&\, B \vdash_{PS^*} B$ [*i.e.*, from $A \,\&\, B$, infer B]

(PS^*5) $\{A, B\} \vdash_{PS^*} A \,\&\, B$ [*i.e.*, from A and B, infer $A \,\&\, B$]

Also, assume that the system (PS^*) has a "deduction theorem" of the following sort:

(PS^*6) If $\Gamma \cup \{A\} \vdash_{PS^*} B$, then $\Gamma \vdash_{PS^*} \sim(A \,\&\, \sim B)$.

Exercise #6. Show that any system (PS^*) with these six properties is *strongly complete* for the standard truth-table semantics for $\sim$ and &. That is, show that every p-consistent [in (PS^*)] set of formulas of P^* has a model, by extending it first to a maximal p-consistent set. In other words, show (Henkin-style) that:

$$\text{If } \Gamma \vDash_{P^*} A, \text{ then } \Gamma \vdash_{PS^*} A,$$

by proving its contrapositive:

$$\text{If } \Gamma \nvdash_{PS^*} A, \text{ then } \Gamma \nvDash_{P^*} A.$$

This will involve showing that

$$\text{If } \Gamma \cup \{\sim A\} \text{ is } p\text{-consistent [in } (PS^*)\text{], then } \Gamma \cup \{\sim A\} \text{ has a model [in } P^*\text{'s semantics].}$$

And, that will involve proving an appropriate version of Lindenbaum's Lemma for (PS^*). You will also need to prove some other metatheoretic lemmas here [like the equivalence of "$\Gamma \nvdash_{PS^*} A$" and "$\Gamma \cup \{\sim A\}$ is p-consistent in (PS^*)"]. But, the two main parts of the proof are (*i*) the Lindenbaum construction for any p-consistent $\Gamma \cup \{\sim A\}$, and (*ii*) the Henkin interpretation for P^*, which leads to a *model* of such a $\Gamma \cup \{\sim A\}$.