Another Argument for $\supset$

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In this note, I will outline an argument for $\supset$, which combines (and refines) arguments from Priest and Gibbard. First, some notation. I will use '$\rightarrow$' for the English indicative conditional, and '$\supset$' for material implication. I will (typically) use English connectives in the meta-theory (sometimes, I will abbreviate the meta-theoretic conditional as '$\Rightarrow$', and sometimes I will use the word "entails", which is meant to be synonymous with the meta-theoretic $\Rightarrow$), and I will assume that our meta-theory is *classical*. I will give an argument for the following meta-theoretic statement (understood as a *schema*, which holds for *any* p/q):

$$p \rightarrow q \text{ if and only if } p \supset q$$
$$[\textit{i.e.}, \; p \rightarrow q \Leftrightarrow p \supset q]$$

This requires establishing the following two meta-theoretic conditionals:

① If $p \rightarrow q$, then $p \supset q$. [*i.e.*, $p \rightarrow q \Rightarrow p \supset q$]

② If $p \supset q$, then $p \rightarrow q$. [*i.e.*, $p \rightarrow q \Leftarrow p \supset q$]

My strategy will be to prove ① first, and then *use* ① to prove ②. Here goes.

Argument for ①. Assuming a classical meta-theory, ① requires *only* the following principle:

($\text{MP}_{\rightarrow}$) If p and $p \rightarrow q$, then q. [*i.e.*, *Modus Ponens* for '$\rightarrow$' *preserves truth.*]

Here is my argument for ①. I will actually prove the *contrapositive* of ①.

1	$p \supset q$ is false.	Assumption (for $\Rightarrow$I)
2	p is true.	From (1), by classical logic.
3	q is false.	From (1), by classical logic.
4	$p \rightarrow q$ is true.	Assumption (for RAA)
5	q is true.	From (2) and (4), by ($\text{MP}_{\rightarrow}$).
6	Contradiction.	From (3), (5).
7	$p \rightarrow q$ is false.	From (4)–(6), by (RAA).
8	$p \supset q$ is false $\Rightarrow$ $p \rightarrow q$ is false.	From (1)–(7), by ($\Rightarrow$I).
9	①	From (8), by $\Rightarrow$ contraposition. □

Thus, assuming a classical meta-theory, *all* we need in order to prove ① is ($\text{MP}_{\rightarrow}$). This shows that ① *is virtually equivalent to the assertion that Modus Ponens is truth-preserving for the indicative conditional.*

Argument for ②. My argument for ② depends on the following six principles:

($\text{EXP}_{\rightarrow}$) If $\ulcorner (p \,\&\, q) \rightarrow r \urcorner$, then $\ulcorner p \rightarrow (q \rightarrow r) \urcorner$. [*i.e.*, *Exportation* for '$\rightarrow$' *preserves truth.*]

① If $p \rightarrow q$, then $p \supset q$.

($\text{AND}_{\rightarrow}$) $\ulcorner (p \,\&\, q) \rightarrow q \urcorner$ is a logical truth.

(LTE) If p is a logical truth, and p entails q, then q is a logical truth.

(SUB) If p' is obtained from p by substitution of logical equivalents (*i.e.*, if p' results from substituting q' for q in p, where $q' \Leftrightarrow q$), then p entails p'.

($\text{SDT}^1_{\supset}$) If $\ulcorner p \supset q \urcorner$ is a logical truth, then p entails q.

Here is my argument for ②. This argument will be *direct*.

1	$\ulcorner (p \,\&\, q) \to q \urcorner$ is a logical truth.	(AND$_\to$)
2	$\ulcorner ((p \supset q) \,\&\, p) \to q \urcorner$ is a logical truth.	From (1), by (SDT$^1_\supset$), (SUB), and (LTE).
3	$\ulcorner (p \supset q) \to (p \to q) \urcorner$ is a logical truth.	From (2), by (EXP$_\to$) and (LTE).
4	$\ulcorner (p \supset q) \supset (p \to q) \urcorner$ is a logical truth.	From (3), by ① and (LTE).
5	②	From (4), by (SDT$^1_\supset$).

Therefore, *the only way* one can resist the conclusion that the English indicative conditional $\to$ is *equivalent* to $\supset$ is to reject some of the following six assumptions (or some other classical inference in the meta-theory):

(MP$_\to$) If p and $p \to q$, then q. [*i.e.*, *Modus Ponens* for '$\to$' *preserves truth*.]

(EXP$_\to$) If $\ulcorner (p \,\&\, q) \to r \urcorner$, then $\ulcorner p \to (q \to r) \urcorner$. [*i.e.*, *Exportation* for '$\to$' *preserves truth*.]

(AND$_\to$) $\ulcorner (p \,\&\, q) \to q \urcorner$ is a logical truth.

(LTE) If p is a logical truth, and p entails q, then q is a logical truth.

(SUB) If p' is obtained from p by substitution of logical equivalents (*i.e.*, if p' results from substituting q' for q in p, where $q' \Leftrightarrow q$), then p entails p'.

(SDT$^1_\supset$) If $\ulcorner p \supset q \urcorner$ is a logical truth, then p entails q.

As we have seen, MacFarlane & Kolodny and McGee reject (MP$_\to$). McGee seems to accept *all the other* assumptions of this argument, whereas MacFarlane & Kolodny also reject (EXP$_\to$). I think MacFarlane & Kolodny accept everything here *except* for (MP$_\to$) and (EXP$_\to$). But, it's not at all obvious to me why someone who's worried about $\supset$ should accept (SDT$^1_\supset$). That places a non-trivial constraint on our meta-theoretic "entailment" and "equivalence" relations, which could "trickle down" to our indicative, *especially* if we were inclined to assume some sort of semantic deduction theorem(s) for our *indicative* conditional as well. Consider:

(SDT$^1_\to$) If $\ulcorner p \to q \urcorner$ is a logical truth, then p entails q.

(SDT$^2_\to$) If p entails q, then $\ulcorner p \to q \urcorner$ is a logical truth.

(SDT$^2_\supset$) If p entails q, then $\ulcorner p \supset q \urcorner$ is a logical truth.

I suspect that (SDT)-type assumptions are doing more work than meets the eye here, since it is easy to tacitly presuppose that (SDT)-type principles hold for *both* $\to$ *and* $\supset$. But, of course, if (SDT) *does* hold for *both* connectives, then we can "prove" a *validity-preserving* rendition of the desired equivalence, *trivially*:

1	$\ulcorner p \supset q \urcorner$ is a logical truth.	Assumption (for $\Rightarrow$I)
2	p entails q.	From (1), by (SDT$^1_\supset$).
3	$\ulcorner p \to q \urcorner$ is a logical truth.	From (2), by (SDT$^2_\to$).
4	$\vDash p \supset q \;\Rightarrow\; \vDash p \to q$	From (1)–(3), by $\Rightarrow$I. □

and

1	$\ulcorner p \to q \urcorner$ is a logical truth.	Assumption (for $\Rightarrow$I)
2	p entails q.	From (1), by (SDT$^1_\to$).
3	$\ulcorner p \supset q \urcorner$ is a logical truth.	From (2), by (SDT$^2_\supset$).
4	$\vDash p \to q \;\Rightarrow\; \vDash p \supset q$	From (1)–(3), by $\Rightarrow$I. □

And, if *these* meta-theoretic claims can be shown, then my conjecture about $\to$ and $\supset$ having similar *validity-preserving* (pure conditional) forms starts to look more plausible. We need to think more about (SDT)'s.