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SEMANTICAL ANALYSIS OF MODAL LOGIC II. NON-NORMAL MODAL PROPOSITIONAL CALCULI

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This paper continues the investigations of Kripke [63]. Not only is Kripke [63] presupposed; the reader is advised to have it on hand for ready reference. The notations and terminology of that paper are used freely; in particular, $P, Q, R, \ldots$ are atomic formulae, and $A, B, C, \ldots$ are arbitrary formulae built from them using the connectives $\wedge$, $\sim$, $\Box$. All propositional calculi in this paper have the same formation rules as those of Kripke [63]. However, they in general will lack the rule of necessitation of Kripke [63] and thus will not be normal. Hence (in our opinion), they are intuitively somewhat unnatural; but nevertheless they have an elegant model theory. Among these systems are, notably, Lewis's S2 and S3; this paper extends the results of Kripke [63] to these and other systems. The results of this paper were announced in Kripke [63, abstract].

All systems considered here will be formulated with axiom schemata; substitution is a derived rule.

1. Generalities; Halldén's property.

By R1, or modus ponens, we mean the rule: $A, A \supset B / B$.

Theorem 1. *Let* PL *be any propositional calculus containing the theorem schema* $A \supset (A \wedge \ldots A)$ *and its converse* $(A \wedge \ldots A) \supset A$. *If modus ponens is an admissible (derivable) rule of* PL, PL *can be recursively axiomatized with modus ponens as sole rule of inference.*

Proof. (Essentially given in Craig [53].) Form PL′ as follows: Axioms: $(A \wedge \ldots A) \supset A$ (any number of conjuncts A), and $B \wedge \ldots B$ (m conjuncts), where m is the Gödel number of a proof of B in PL. R1 is the only rule. Then clearly PL contains PL′, and PL′ contains PL because if B is any theorem of PL and m is the Gödel number of a proof thereof, $B \wedge \ldots B$ and $B \wedge \ldots B \cdot\supset\cdot B$ (m conjuncts) are both provable in PL′, whence B is by R1. Q.E.D.

S2, S3, and most systems considered below have an "incompleteness" property of Halldén [51]: some formula $A \vee B$ is derivable, where A and B have no propositional variables in common and neither A nor B is derivable. Halldén essentially showed that no system with his property can possess a normal characteristic matrix.

Theorem 2. *Let* PL *be a propositional calculus containing all classical tautologies and with* R1 *as sole postulated rule. Let A and B be formulae with no propositional variables in common. Suppose $A \vee B$ is provable, and let* PL′ (PL″) *be formed by adding all substitution instances of A (of B) as a new axiom schema to* PL. *Then* PL = PL′ ∩ PL″.

Proof. Let C be a theorem of PL′ and of PL″. We need to derive it in PL. Since $\vdash C$ in PL′, there are finitely many instances $A_1, \ldots, A_m$ of A from which C is derivable in PL′. Hence, by the deduction theorem, $\vdash A_1 \wedge \ldots A_m \cdot\supset\cdot C$ in PL. Similarly, $\vdash B_1 \wedge \ldots B_n \cdot\supset\cdot C$ in PL for certain substitution instances $B_1, \ldots, B_n$ of B. So we need only derive $A_1 \wedge \ldots A_m \cdot\vee\cdot B_1 \wedge \ldots B_n$ in PL. By relettering B if necessary, we can assume that $A_1, \ldots, A_m$, as well as A, contain no variables in common with B; no matter how B is thus relettered, we can still derive $A \vee B$ in PL. Since A and B have no propositional variables in common, by substitution, we get $\vdash A_i \vee B$ in PL ($i = l, \ldots, m$); hence $\vdash A_1 \wedge \ldots A_m \cdot\vee\cdot B$. Since in this latter formula the disjuncts still have no variable in common, substituting again we get $\vdash (A_1 \wedge \ldots A_m) \vee B_j$ ($j = l, \ldots, n$), and hence $\vdash (A_1 \wedge \ldots A_m) \vee (B_1 \wedge \ldots B_n)$. Q.E.D.

Theorem 3. *If* PL *is a propositional calculus, it possesses a normal characteristic matrix if and only if the following four conditions hold:*

(*a*) PL *is consistent* (*i.e.*, $\dashv A \wedge \sim A$).
(*b*) R1 *is an admissible rule of* PL.
(*c*) PL *contains all classical tautologies.*
(*d*) PL *does not possess Halldén's property.*

The necessity of the conditions is clear. The proof of sufficiency, which we omit, proceeds by constructing a normal characteristic matrix by Lindenbaum's method.

Although we are assuming that all propositional calculi have the formation rules of Kripke [63], it is clear that the theorems above hold for a wider class of formation rules. Further, Theorem 2 applies, e.g., to systems containing the Hilbert-Bernays positive logic (in place of classical logic).

2. Propositional calculi considered. All systems considered contain all tautologies of classical propositional logic as axioms; thus these axioms will not be listed explicitly [1].

The system E2 of Lemmon [57] is characterized by the axioms:

A1. $\Box A \supset A$,

A3. $\Box (A \supset B) \cdot\supset\cdot \Box A \supset \Box B$,

and the rules R1 and

(Eb) If $\vdash A \supset B$, then $\vdash \Box A \supset \Box B$.

We get a system E3 if we replace A3 in E2 by the stronger axiom:

(1) $\Box (A \supset B) \supset \Box (\Box A \supset \Box B)$.

It is easy to show that (Eb) is equivalent to $A \supset B / \Diamond A \supset \Diamond B$.

Now by Theorem 1, E2 and E3 can be reformulated with R1 as sole rule of inference. Alternatively, we can get more natural reaxiomatizations of these two systems by dropping the rule (Eb), and replacing it by the clause: if B is an axiom and A is any formula, $\Box A \supset \Box B$ is an axiom. Then, with R1 as sole postulated rule, we can obtain $B / \Box A \supset \Box B$ as a derived rule, and thence (Eb) (by the inference: $A \supset B / \Box A \supset \Box (A \supset B)$, hence $\Box A \supset \Box B$ by A3 and truth-functional logic) [2]. Given a formulation of E2 (E3) with R1 as sole (primitive) rule, define S2 (S3) as the result of adding to this formulation the schema $\Box (A \supset A)$. Further, for a formulation of E2 with R1 as sole rule, define $S2^n$ by adding thereto:

(2) $\Box^n (A \supset A) \qquad (0 \leqslant n < \infty)$.

[1] This proviso was inadvertently neglected in Kripke [63]. In connection with Kripke [63], Footnote 2 on p. 69, we should mention that an outline of Hintikka's work has since appeared as Hintikka [61]; and that the "pupils" of T. J. Smiley mentioned in that footnote can now be more accurately specified as Francis Drake.

[2] We can prove that if B is derivable, so is $\Box A \supset \Box B$, by induction on the length of a proof of B in the reaxiomatized version of E2 (E3). If B is an axiom, the desired result is clear. If B is derived from C and $C \supset B$ by R1, we can assume by the inductive hypothesis that $\Box A \supset \Box C$ and $\Box A \supset \Box (C \supset B)$ are derivable. By axiom A3 and truth-functional logic, we obtain $\vdash \Box A \supset (\Box C \supset \Box B)$ from the second of these; and from this together with the first we get the desired $\vdash \Box A \supset \Box B$.

Note that the rule $B / \Box A \supset \Box B$ is a trivial consequence of (Eb) (since $A \supset B$ is derivable from B). So the reaxiomatized version of E2 (E3) is clearly *contained* in E2 (E3). Above we indicated how to show that it *contains* E2 (E3).

Here $\Box^n$ indicates a sequence of n necessity signs. Thus $S2^0$ is E2; $S2^1$ is S2.

Now if A is a theorem of E2 (E3), so is $A \supset A \cdot\supset\cdot A$, hence by (Eb) $\vdash \Box (A \supset A) \supset \Box A$. Hence $\vdash \Box A$ in S2 (S3). Similarly (using n applications of (Eb)), $\Box^n (A \supset A) \supset \Box^n A$ is a theorem of E2, so by R1 $\Box^n A$ is a theorem of $S2^n$. In Theorem 6 below we will show conversely that if $\vdash \Box^n A$ in $S2^n$, then $\vdash A$ in E2.

If we define $S2^\infty$ by adding (2) for all n as schemata to a formulation of E2 with R1 as sole rule, it is easily shown that if B is an axiom of $S2^\infty$, $\vdash \Box B$ in $S2^\infty$ (for every axiom is either $\Box^n (A \supset A)$ or a theorem of E2). Hence by induction on the length of a proof of B in $S2^\infty$,[3] $\vdash B$ in $S2^\infty$ implies $\vdash \Box B$; and thus $S2^\infty$ is equivalent to Feys-von Wright's M (cf. Kripke [63]).

Notice that in E3, since $\vdash \Box B \supset \Box (A \supset B)$, by two applications of (1) to the consequent, we get $\vdash \Box B \supset \Box (\Box \Box A \supset \Box \Box B)$, and hence $\vdash \Box \Box A \cdot\supset\cdot \Box B \supset \Box \Box B$. Hence it can be shown that, if we defined $S3^2$ by adding (1) to $S2^2$, $S3^2$ would collapse into S4. (Some intuitive motivation for the facts that $S3^2 = S4$, $S2^\infty = M$ will become apparent from the model theory below.)

The systems $S2^n$, E3, and S3 will be discussed, model-theoretically, below. In the rest of this section, we mention other systems, and their relations to the previous systems and to each other. The Falsum system is the system with R1 as sole rule and the following single axiom:

(3) $\sim \Box A$.

Axiom (3) reduces "$\Box$" to a *falsum* operator. It can easily be verified that E2 and E3 admit an interpretation with "$\Box$" as the *falsum* operator. So does Lemmon's E4, whose rules are R1 and (Eb) and whose axioms are A1, A3, and

A4. $\Box A \supset \Box \Box A$.

Lemmon's E5 collapses into S5; the present writer has proposed instead a system E5 defined by A1, A3,

A2. $\Box A \cdot\supset\cdot \sim \Box B \supset \Box \sim \Box B$,

[3] We have just handled the case where B is an axiom. If B is derived by R1 from C and $C \supset B$, we can assume by inductive hypothesis that $\Box C$ and $\Box (C \supset B)$ are derivable. From the second of these we get by A3 $\vdash \Box C \supset \Box B$, hence using $\Box C$ and R1, we obtain the desired $\vdash \Box B$.

and R1 and (Eb). This system is still contained in the Falsum system.

Now since $\vdash B \cdot \supset \cdot A \supset A$, by (Eb) $\vdash \Box B \supset \Box (A \supset A)$ in E2; *i.e.*, $\vdash \Box (A \supset A) \vee \sim \Box B$. Hence, applying Theorem 2 to E2 (E3) reformulated with R1 as sole rule, E2 (E3) is the intersection of S2 (S3) with the Falsum system. Theorem 2 further implies that E4 (E5) is analogously related to S4 (S5).

We remark that all systems so far considered (except the Falsum) are contained in the Trivial system, which is based on the axioms $\Box A \supset A$ and $A \supset \Box A$. R1 is the only rule.

The Łukasiewicz Ł-modal system is axiomatized by the axioms A1 and

(4) $A \supset B \cdot \supset \cdot \Box A \supset \Box B$.

R1 is the only rule (cf. Łukasiewicz [53]). Curry's Gentzen rules for possibility in Curry [52] turn out to be equivalent to the Ł-modal system, although by the time Curry [52] was published, he no longer advocated this system as natural. Another formulation of the system has been rediscovered by Jean Porte. The Ł-modal system is contained in both the Trivial and the Falsum systems. In fact, we easily derive, from $\vdash B \supset \cdot A \supset B$ and (4), $\vdash B \cdot \supset \cdot \Box A \supset \Box B$, hence $\vdash \sim \Box A \cdot \vee \cdot B \supset \Box B$. So by Theorem 2, the Ł-modal system is the intersection of the Trivial and the Falsum systems, a fact first proved by Prior [57]. Thus the Ł-modal system contains all the E-systems.

3. Models. Any attempt to develop a model theory for E2 and E3 must face the problem that no formula of the form $\Box B$ is provable in E2 or E3, not even $\Box (A \supset A)$. If, as in Kripke [63], however, we were to define a formula $\Box B$ as true in a "world" $\mathbf{H}_1$ iff B is true in every $\mathbf{H}_2$ "possible relative to" $\mathbf{H}_1$, and evaluated the truth-functions by truth-tables, then since $A \supset A$ would clearly be true in any world $\mathbf{H}_2$, $\Box (A \supset A)$ would have to be true in any world $\mathbf{H}_1$. Similar difficulties arise in S2 and S3 (in S2^n), owing to the fact that $\Box^2(A \supset A)(\Box^n(A \supset A))$ is unprovable.

We observe, however, that $\vdash \Box B \supset \Box (A \supset A)$ in E2, and hence $\vdash \sim \Box (A \supset A) \supset \sim \Box B$; if $A \supset A$ is not necessary, then nothing is. This leads us to divide the "possible worlds" into two classes; the "normal worlds", in which necessity is evaluated according to the Leibnitzian prescription of Kripke [63]; and the non-normal worlds, in which $\Box B$ is always false.

We define, then, a *model structure* (m.s.) to be an ordered triple $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, where $\mathbf{K}$ is a set, $\mathbf{G} \in \mathbf{K}$, and $\boldsymbol{R}$ is a relation defined on $\mathbf{K}$ with the following *quasi-reflexivity* property: If $\mathbf{H}_1 \boldsymbol{R} \mathbf{H}_2$ ($\mathbf{H}_1, \mathbf{H}_2 \in \mathbf{K}$), then $\mathbf{H}_1 \boldsymbol{R} \mathbf{H}_1$. An element of $\mathbf{K}$ (a "world") $\mathbf{H}$ such that $\mathbf{H} \boldsymbol{R} \mathbf{H}$ is called *normal*.

(Alternatively, we could define a model structure as a quadruple $(\mathbf{G}, \mathbf{K}, \boldsymbol{R}, \mathbf{N})$, where $\mathbf{K}$ is a set, $\mathbf{G} \in \mathbf{K}$, $\mathbf{N} \subseteq \mathbf{K}$, and $\boldsymbol{R}$ is a relation on $\mathbf{K}$ which is reflexive on $\mathbf{N}$. The normal worlds are the elements of $\mathbf{N}$. This definition leads to a slightly more general class of models at the expense of one more primitive; its advantages will become apparent in Section 7.)

A m.s. is also called an E2 m.s. If $\boldsymbol{R}$ is transitive, a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is an E3 m.s. An E2 (E3) m.s. in which $\mathbf{G}$ is normal is called an S2 (S3) m.s.

Temporarily let Q be the following relation between elements of $\mathbf{K}$ and non-negative integers: $\mathbf{H} \, Q \, n$ iff either $n=0$ and $\mathbf{H}=\mathbf{G}$ or $n \geqslant 1$ and there exist $\mathbf{H}_0, \ldots, \mathbf{H}_n \in \mathbf{K}$ such that $\mathbf{H}_0 = \mathbf{G}$, $\mathbf{H}_n = \mathbf{H}$, and $\mathbf{H}_i \boldsymbol{R} \mathbf{H}_{i+1}$ $(0 \leqslant i < n)$. Define the *depth* n of an element $\mathbf{H}$ of $\mathbf{K}$ to be the least n such that $\mathbf{H} \, Q \, n$; if no such n exists, we say the depth is *infinite*.

We define an E2 m.s. to be an S2^n m.s. iff every $\mathbf{H} \in \mathbf{K}$ of depth $<n$ is normal. For $n=0$ $(n=1)$, this coincides with the previous notion of E2 (S2) m.s.

An E2 (E3, S2^n, S3) *model* φ for a wff A of E2 (E3, S2^n, S3) is a binary function $\varphi(P, \mathbf{H})$ associated with a given E2 (E3, S2^n, S3) m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$. As in Kripke [63], the first variable P ranges over atomic subformulae of A, while the second variable $\mathbf{H}$ ranges over elements of $\mathbf{K}$. The range of φ is the set $\{\mathbf{T}, \mathbf{F}\}$.

Given a model φ associated with a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, we define for any subformula B of A a value $\varphi(B, \mathbf{H})$. If B is atomic, the value has already been given. If $\varphi(B, \mathbf{H})$ and $\varphi(C, \mathbf{H})$ have been defined, for each $\mathbf{H}$, then define $\varphi(B \wedge C, \mathbf{H})$, $\varphi(\sim B, \mathbf{H})$ as in Kripke [63]. We define $\varphi(\Box B, \mathbf{H}) = \mathbf{T}$ iff $\mathbf{H}$ is normal and $\varphi(B, \mathbf{H}') = \mathbf{T}$ for all $\mathbf{H}'$ such that $\mathbf{H} \boldsymbol{R} \mathbf{H}'$; otherwise, $\varphi(\Box B, \mathbf{H}) = \mathbf{F}$. (Hence, in particular, $\varphi(\Box B, \mathbf{H}) = \mathbf{F}$ automatically, if $\mathbf{H}$ is not normal.)

As in Kripke [63], a formula A is true in a model φ associated with a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ iff $\varphi(A, \mathbf{G}) = \mathbf{T}$. A is valid (for S2^n, E3, S3) iff it is true in every (S2^n, E3, S3) model. Notice that here, unlike the situation in Kripke [63], the distinguished element $\mathbf{G}$ in a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ plays an essential role in the definitions. For example, in an S2 model φ on a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, $\varphi(\Box(A \supset A), \mathbf{G})$ must equal $\mathbf{T}$, since $\mathbf{G}$ is normal in an S2 m.s.; hence $\Box (A \supset A)$ is valid in S2.

We can define *connected* models and *tree* models as in Kripke [63]. As in Kripke [63], we could confine ourselves to connected models without essentially changing the theory. Notice that, in a connected m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, every $\mathbf{H} \in \mathbf{K}$ has a finite depth; and that, in a connected m.s. where $\boldsymbol{R}$ is transitive, every $\mathbf{H} \in \mathbf{K}$ has depth $\leqslant 1$. Hence, if we define a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ to be an $S2^{\infty}$ m.s. if every $\mathbf{H} \in \mathbf{K}$ of finite depth is normal, then a connected $S2^{\infty}$ m.s. is an M m.s. (cf. Kripke [63]). If we define a $S3^2$ m.s. as an E3 m.s. where every $\mathbf{H} \in \mathbf{K}$ of depth <2 is normal, then a connected $S3^2$ m.s. is an S4 m.s. Conversely, under these definitions, an M (S4) m.s. is an $S2^{\infty}$ ($S3^2$) m.s. These facts provide the model-theoretic motivation for the results in Section 2 that $S2^{\infty} = M$, $S3^2 = S4$.

4. Semantic tableaux. We can adopt almost all of the notions of Kripke [63] for tableau constructions without change. Again, we have, at a given stage of the construction, a system of alternative sets of tableaux. Each alternative set $\mathscr{S}$ is ordered by a relation R which now need be only quasi-reflexive. The ordered set $\mathscr{S}$ can, in a natural way, be given the form of a tree; the origin $\mathbf{t}_0$ of the tree is the (descendant in $\mathscr{S}$ of) the main tableau. Replacing $\mathbf{G}$ by $\mathbf{t}_0$, we can define the depth n of a tableau $\mathbf{t}$ in $\mathscr{S}$, just as it was previously defined for a world: the depth n of $\mathbf{t}$ is the least n such that either $n=0$ and $\mathbf{t}=\mathbf{t}_0$, or $n \geqslant 1$ and there exist $\mathbf{t}_1, \ldots, \mathbf{t}_n$ such that $\mathbf{t}_n = \mathbf{t}$, $\mathbf{t}_i \, R \, \mathbf{t}_{i+1} (0 \leqslant i < n)$.

Now a tableau $\mathbf{t}$ of an alternative set $\mathscr{S}$ is called *normal* iff $\mathbf{t} \, R \, \mathbf{t}$. We require R to be quasi-reflexive: $\mathbf{t}_1 \, R \, \mathbf{t}_2$ implies $\mathbf{t}_1 \, R \, \mathbf{t}_1$. In E3 and S3 constructions, we further require that R be transitive. In S2 and S3 constructions, the main tableau $\mathbf{t}_0$ must be normal. In $S2^n$ constructions any tableau of depth $<n$ is normal. Further, for all systems we require that a tableau with a formula $\Box$ B on the left be normal (since, correspondingly, $\varphi(\Box B, \mathbf{H}) = \mathbf{T}$ implies that $\mathbf{H}$ is normal). In short, as in Kripke [63], the properties assumed for R parallel those of $\boldsymbol{R}$; and, as in Kripke [63], the relation R is to hold in a set only if the preceding requires it to hold.

All rules except Yr can be adopted without change [4] from Kripke [63]. Yr now reads:

Yr. If $\Box A$ appears on the right of a normal tableau $\mathbf{t}$, we start out a new tableau $\mathbf{t}'$, with A on the right, and such that $\mathbf{t} \, R \, \mathbf{t}'$.

[4] For $\varLambda$r, the italicized condition in Kripke [63], p. 73, included there to ensure reflexivity of R, should be dropped in favor of: "*Further if* $\mathbf{t} \, R \, \mathbf{t}$, *we stipulate that* $\mathbf{t}_1 \, R \, \mathbf{t}_1$."

Notice that Yr, as stated, actually renders the "requirement" of quasi-reflexivity superfluous. For if $\mathbf{t} \, R \, \mathbf{t}$ is not required for some reason other than quasi-reflexivity (e.g., occurrence of some formula $\Box B$ on the left, or being of depth $<n$ in an $S2^n$ construction), then Yr will forever be inapplicable to $\mathbf{t}$, so that no tableau $\mathbf{t}'$ with $\mathbf{t} \, R \, \mathbf{t}'$ will ever be introduced, and thus quasi-reflexivity will hold vacuously. Hence, in an E2 or E3 (an S3, an $S2^n$) construction, a tableau $\mathbf{t}$ will be normal iff it has some formula $\Box B$ on the left (either has some formula $\Box B$ on the left or is the main tableau, either has some formula $\Box B$ on the left or is of depth $<n$).

Theorem 4. *A is valid in* E2 (E3, S3, $S2^n$) *iff the construction for A is closed* (*in the appropriate system*).

The proof of this theorem is exactly like that of the corresponding theorem in Kripke [63], Section 3.2.

4.1. Trees and a reformulation of the rules. As in Kripke [63], each of the alternative sets of a stage of a construction can be given the form of a tree. In fact, if, following Kripke [63], we define $\mathbf{t}_1 \, S \, \mathbf{t}_2$ to hold iff $\mathbf{t}_1$ and $\mathbf{t}_2$ are descendants in $\mathscr{S}$ of tableaux $\mathbf{t}_1'$ and $\mathbf{t}_2'$, such that $\mathbf{t}_2'$ was introduced by an application of Yr to $\mathbf{t}_1'$, then S is a tree relation on $\mathscr{S}$. Following Kripke [63], Section 3.3., we raise the question whether the rules can be reformulated in terms of S, in such a way that a rule applied to a tableau $\mathbf{t}$ can affect at most $\mathbf{t}$ and tableaux contiguous to $\mathbf{t}$ (Kripke [63], p. 81). Notice that, if we replace R by S, we can define the depth of a tableau $\mathbf{t}$ in terms of S, rather than R; for $S2^n$ constructions this notion of depth will coincide with the original one (but for E3 and S3, where R is transitive, it will not). Now, in terms of S, we can define a tableau $\mathbf{t}$ of an $S2^n$ construction to be normal iff it contains a formula $\Box A$ on the left or is of depth $<n$, for an E3 construction iff it contains a formula $\Box A$ on the left, and for an S3 construction iff it contains a formula $\Box A$ on the left or is the main tableau of its alternative set. We can then reformulate all rules except Y1 simply by replacing R by S (for $\varLambda$r we drop the condition intended to ensure quasi-reflexivity; cf. Kripke [63], p. 80). For $S2^n$, Y1 now becomes like Y1 for M in Kripke [63], p. 81:

Y1. Let $\Box A$ appear on the left of $\mathbf{t}$. Then put A on the left of $\mathbf{t}$ and of every tableau $\mathbf{t}'$ such that $\mathbf{t} \, S \, \mathbf{t}'$.

For S3 and E3, we need to reformulate Y1 so as to obtain a surrogate for the transitivity property of R. We base our reformulation on the following fact, easily verified for a model φ on an E3 m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$: If $\varphi(\Box A, \mathbf{H}_1) = \mathbf{T}$, $\mathbf{H}_1 \in \mathbf{K}$, $\mathbf{H}_2$ is a normal element of $\mathbf{K}$, and $\mathbf{H}_1 \, R \, \mathbf{H}_2$, then $\varphi(\Box A, \mathbf{H}_2) = \mathbf{T}$. So Y1 becomes:

Y1. Let $\Box A$ appear on the left of a tableau $\mathbf{t}$. Then: (1) put A on the left of $\mathbf{t}$; (2) put A on the left of every tableau $\mathbf{t}'$ such that $\mathbf{t} \, S \, \mathbf{t}'$; (3) put $\Box A$ on the left of every *normal* tableau $\mathbf{t}'$ such that $\mathbf{t} \, S \, \mathbf{t}'$.

Notice that if $\Box A$ appears on the left of $\mathbf{t}$ and $\mathbf{t} \, S \, \mathbf{t}'$, $\mathbf{t}' \, S \, \mathbf{t}''$, then $\mathbf{t}'$ must be normal (otherwise it could not be a descendant of a tableau to which Yr was applied, so that $\mathbf{t}' \, S \, \mathbf{t}''$ would be impossible). Hence by Y1, we put $\Box A$ on the left of $\mathbf{t}'$, and thus we put A on the left of $\mathbf{t}''$. This process can be iterated an arbitrary number of times, so we get the *effect* of transitivity (cf. the analogous discussion for S4, Kripke [63], p. 81).

The equivalence of the "S-formulation" to the "R-formulation" can be proved without much trouble. The "S-formulation" will be used throughout Section 5.

5. Completeness theorem.

5.1. Consistency property. We prove the completeness theorem for all systems $S2^n$ $(0 \leqslant n < \infty)$ and for E3 and S3. We first need to show that every provable formula of a given system is valid (in the appropriate model theory). For E2 and E3, we can verify the validity of the axiom schemata directly, and then verify the rules R1 and (Eb). R1 is trivial. To verify (Eb), suppose we are given an E2 (E3) m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, and an associated model φ such that $\varphi(\Box A \supset \Box B, \mathbf{G}) = \mathbf{F}$, so that $\Box A \supset \Box B$ is not valid in E2 (E3); we need to verify that $A \supset B$ is not valid in E2 (E3). Since $\varphi(\Box A \supset \Box B, \mathbf{G}) = \mathbf{F}$, $\varphi(\Box A, \mathbf{G}) = \mathbf{T}$, and hence $\mathbf{G}$ is normal. Hence, since $\varphi(\Box B, \mathbf{G}) = \mathbf{F}$, there exists $\mathbf{H}_0 \in \mathbf{K}$ such that $\varphi(B, \mathbf{H}_0) = \mathbf{F}$ and $\mathbf{G} \, \boldsymbol{R} \, \mathbf{H}_0$. Since $\mathbf{G} \, \boldsymbol{R} \, \mathbf{H}_0$ and $\varphi(\Box A, \mathbf{G}) = \mathbf{T}$, $\varphi(A, \mathbf{H}_0) = \mathbf{T}$. So $\varphi(A \supset B, \mathbf{H}_0) = \mathbf{F}$. Now $(\mathbf{H}_0, \mathbf{K}, \boldsymbol{R})$ is an E2 (E3) m.s., and we define a model $\varphi'(P, \mathbf{H})$ thereon by stipulating that $\varphi'(P, \mathbf{H}) = \varphi(P, \mathbf{H})$ for all $\mathbf{H} \in \mathbf{K}$. That is, φ and φ' are the same binary *functions*, but one is associated with $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ and the other with $(\mathbf{H}_0, \mathbf{K}, \boldsymbol{R})$. Hence, since $\varphi(A \supset B, \mathbf{H}_0) = \mathbf{F}$, $\varphi'(A \supset B, \mathbf{H}_0) = \mathbf{F}$, and φ' falsifies $A \supset B$, Q.E.D. So we have now verified that all theorems of E2 (E3) are valid in E2 (E3). In $S2^n$, all axioms are theorems of E2 (hence valid in E2 and *a fortiori* in $S2^n$) or are of the form $\Box^n(A \supset A)$. The latter are easily verified in $S2^n$ to be valid on account of the restriction (in an $S2^n$ m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$) that every $\mathbf{H} \in \mathbf{K}$ of depth $< n$ be normal. The rule R1, which is the only rule of $S2^n$, is verified trivially. Similarly, for S3, $\Box(A \supset A)$ is obviously valid, since $\mathbf{G}$ must be normal in an S3 m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$; and the other verifications are trivial.

5.2. Completeness property. As in Kripke [63], we prove that if the construction based on S for A is closed, A is valid. The rank of a tableau, the associated and the characteristic formula, are as in Kripke [63]. As in Kripke [63], we need to prove the following:

Lemma. *If A_0 is the characteristic formula of the initial stage of an $S2^n$ (E3 or S3) construction, and B_0 is the characteristic formula of any stage, then* $\vdash A_0 \supset B_0$ in E2 (E3).

Proof. As in Kripke [63], we prove that the characteristic formula of any stage A_m implies the characteristic formula A_{m+1} of the next stage. In fact, we prove that, in any $S2^n$ construction, $\vdash A_m \supset A_{m+1}$ is already provable in E2; and similarly that, for an E3 or S3 construction, $\vdash A_m \supset A_{m+1}$ is already provable in E3. We can repeat all the "preliminary observations" in the proof of the corresponding lemma in Kripke [63], noting that the inference from $\vdash X \supset Y$ to $\vdash \Diamond X \supset \Diamond Y$ (Kripke [63], p. 84, lines 13–16), which was justified in Kripke [63] by an appeal to the rule R2, is justified in E2 and E3 by (Eb). And in fact, all the cases considered in Kripke [63] except Y1 can be repeated here without change. (It is notable that the revision of Yr makes no difference to the verification of the case Yr – in fact, the case would go through even if the applicability of Yr were not restricted to normal tableaux.) So we verify only Y1:

Case Y1. In $S2^n$, we have $\Box A$ on the left of a tableau $\mathbf{t}$, and we are required to put A on the left of $\mathbf{t}$ and of every tableau $\mathbf{t}'$ such that $\mathbf{t} \, S \, \mathbf{t}'$. As in Kripke [63], putting A on the left of $\mathbf{t}$ itself is simply justified by the axiom $\Box A \supset A$. To justify putting A on the left of $\mathbf{t}'$, where $\mathbf{t} \, S \, \mathbf{t}'$, observe (as in Kripke [63]) that if the characteristic formula of $\mathbf{t}$ is $X \wedge \Box A \wedge \Diamond B$, where B is the characteristic formula of $\mathbf{t}'$, putting A on the left of $\mathbf{t}'$ changes this characteristic formula of $\mathbf{t}$ to $X \wedge \Box A \wedge \Diamond (B \wedge A)$. We can justify this transformation (as in Kripke [63]) if we can prove $\Box A \wedge \Diamond B . \supset . \Diamond (B \wedge A)$ in E2. Well, we certainly have $\vdash A \supset (B \cdot\supset\cdot B \wedge A)$, so by (Eb) we get $\vdash \Box A \supset \Box (B \cdot\supset\cdot B \wedge A)$. But by the axiom A3, restated in terms of possibility, $\vdash \Box (B \cdot\supset\cdot B \wedge A) \cdot\supset\cdot \Diamond B \supset \Diamond (B \wedge A)$, so hence $\vdash \Box A \cdot\supset\cdot \Diamond B \supset \Diamond (B \wedge A)$, which reduces to the desired $\vdash \Box A \wedge \Diamond B \cdot\supset\cdot \Diamond (B \wedge A)$. Now for E3 and S3, we need to verify further that it is legitimate to put $\Box A$ on the left of every normal tableau $\mathbf{t}'$ such that $\mathbf{t}\, S\, \mathbf{t}'$. Now if $\mathbf{t}\, S\, \mathbf{t}'$, $\mathbf{t}'$ can be normal only if it has some formula $\Box C$ on the left. (In S3, the main tableau $\mathbf{t}_0$ of a set is stipulated to be normal even if it lacks such a formula; but the condition $\mathbf{t}\, S\, \mathbf{t}'$ clearly rules out the possibility that $\mathbf{t}'$ can be the main tableau, i.e., the origin of the tree structure.) So we can assume that the characteristic formula B of $\mathbf{t}'$ has the form $B' \wedge \Box C$. Thus we can justify putting $\Box A$ on the left of $\mathbf{t}'$ if we can prove $\Box A \wedge \Diamond (B' \wedge \Box C) \cdot\supset\cdot \Diamond (B' \wedge \Box C \wedge \Box A)$ in E3. Now we have $\vdash A \cdot\supset\cdot C \supset A$, and hence by (Eb), $\vdash \Box A \supset \Box (C \supset A)$ in E3. But, by the characteristic axiom of E3, $\vdash \Box (C \supset A) \supset \Box (\Box C \supset \Box A)$. Further, it is easy to prove that $\vdash \Box (\Box C \supset \Box A) \cdot\supset\cdot \Diamond (B' \wedge \Box C) \supset \Diamond (B' \wedge \Box C \wedge \Box A)$. Combining these results, by the transitivity of $\supset$ we get $\vdash \Box A \cdot\supset\cdot \Diamond (B' \wedge \Box C) \supset \supset \Diamond (B' \wedge \Box C \wedge \Box A)$, which is easily transformed into the desired result. Q.E.D.

Theorem 5. *If A is valid, then A is provable.*

Proof. The theorem, of course, is asserted relative to a particular system. We prove it first for E2 and E3. If A is valid, then the construction for A is closed. Hence there is a stage of the construction such that each alternative set at this stage contains a closed tableau. Let $D_1 \vee \ldots D_m$ be the characteristic formula of the stage, where the D_j $(j=1, \ldots, m)$ are the characteristic formulae of the alternative sets of the stage. We prove $\vdash \sim D_j$ for each j. Now D_j is the characteristic formula of an alternative set $\mathscr{S}$ which contains a closed tableau $\mathbf{t}$. Now let $\mathbf{t}_0$ be the main tableau of the alternative set $\mathscr{S}$. Since $\mathscr{S}$ has the structure of a tree, there is a unique branch of the tree going from $\mathbf{t}_0$ to $\mathbf{t}$. That is to say, there are tableaux $\mathbf{t}_1, \ldots, \mathbf{t}_p$ such that $\mathbf{t}_p = \mathbf{t}$ and $\mathbf{t}_i\, S\, \mathbf{t}_{i+1}$ $(i=0, \ldots, p-1)$. (We intend this assertion to allow the possibility $\mathbf{t} = \mathbf{t}_p = \mathbf{t}_0$, $p=0$.) We assert that, if X_i is the characteristic formula of $\mathbf{t}_i$, then $\vdash \sim X_i$. Since the characteristic formula of a set is defined as identical with the characteristic formula of its main tableau, X_0 is identical with D_j, and the assertion implies $\vdash \sim D_j$. To prove the assertion, we show that $\vdash \sim X_p$, and that for $i \geqslant 1$, $\vdash \sim X_i$ implies $\vdash \sim X_{i-1}$. Now X_p, being the characteristic formula of $\mathbf{t}$, has the form $Y \wedge B \wedge \sim B$, where B occurs on both the left and the right of $\mathbf{t}\ (= \mathbf{t}_p)$, so clearly $\vdash \sim X_p$. Assuming $\vdash \sim X_i$, if $i \geqslant 1$, we have $\mathbf{t}_{i-1}\, S\, \mathbf{t}_i$. Since $\mathbf{t}_{i-1}\, S\, \mathbf{t}_i$, the tableau $\mathbf{t}_{i-1}$ must be normal; for otherwise the rule Yr would never have been applied to it, contrary to the fact that $\mathbf{t}_{i-1}\, S\, \mathbf{t}_i$. So (remembering that we are dealing with E2 or E3), $\mathbf{t}_{i-1}$ has some formula $\Box B$ on the left. Hence its characteristic formula X_{i-1} has the form $Y \wedge \Box B \wedge \Diamond X_i$. But since $\vdash \sim X_i$, *a fortiori* $\vdash B \supset \sim X_i$, hence by (Eb), $\vdash \Box B \supset \Box \sim X_i$, and hence $\vdash \sim (\Box B \wedge \Diamond X_i)$. This implies $\vdash \sim X_{i-1}$, as required.

So we have proved $\vdash \sim D_j$ for each j. Hence we obtain $\vdash \sim (D_1 \vee \ldots D_n)$. But, by the lemma just proved, since the characteristic formula of the initial stage is $\sim A$, $\vdash \sim A \supset (D_1 \vee \ldots D_m)$. Hence, combining these results, $\vdash A$.

This proof worked for E2 and E3. What about S2^n and S3? We treat S2, S3, and S2^2; the treatment of S2^n for arbitrary n is similar. In S2 and S3, we can no longer assert that a normal tableau must have some formula $\Box B$ on the left; the main tableau $\mathbf{t}_0$ is an exception. But the axiom $\Box (C \supset C)$ of S2 and S3 shows that, for any formula Y, $\vdash Y \cdot\supset\cdot \Box (C \supset C) \wedge Y$. By contraposition, $\vdash \sim (Y \wedge \Box (C \supset C)) \supset \sim Y$ in S2 and S3. Thus if Y is X_0, the preceding proof certainly shows that $\vdash \sim (\Box (C \supset C) \wedge X_0)$ even in E2 (E3); but then, by modus ponens, we can derive $\vdash \sim X_0$ in S2 (S3). Similarly in S2^2, we can no longer be sure that $\mathbf{t}_0$ and $\mathbf{t}_1$ contain a formula $\Box B$ on the left, even if both are normal, although we can be sure of this for any tableau of depth $\geqslant 2$. But the characteristic formula X_0 of $\mathbf{t}_0$ has the form $Y \wedge \Diamond X_1$. Using the axiom $\Box\Box (C \supset C)$ of S2^2, we get $\vdash Y \wedge \Diamond X_1 \cdot\supset\cdot Y \wedge \Box (C \supset C) \wedge \Diamond (\Box (C \supset C) \wedge X_1)$. (If $p=0$, X_1 drops out, and we simply use $\vdash Y \cdot\supset\cdot Y \wedge \Box (C \supset C)$.) Now, we can infer (by the preceding reasoning), successively in E2, $\sim (\Box (C \supset C) \wedge X_1)$, and then $\sim (Y \wedge \Box (C \supset C) \wedge \Diamond (\Box (C \supset C) \wedge X_1))$. Combining this with the previous result, we get $\vdash \sim (Y \wedge \Diamond X_1)$, i.e., $\vdash \sim X_0$. Similarly for S2^n; and then the reasoning goes through as before. Q.E.D.

6. Applications.

Theorem 6. *$\vdash A$ in* E2 (E3) *if and only if* $\vdash \Box^n A$ $(\vdash \Box A)$ *in* S2^n (S3).

Proof. The "only if" part has been proved above. For the "if" part, observe, e.g., for S2, that if A is not derivable in E2, then there is an E2 countermodel φ on a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ for it. Define a m.s. $(\mathbf{G}', \mathbf{K}', \boldsymbol{R}')$, where $\mathbf{K}' = \mathbf{K} \cup \{\mathbf{G}'\}$, $\mathbf{G}' \notin \mathbf{K}'$, and, for $\mathbf{H}_1'$ and $\mathbf{H}_2' \in \mathbf{K}'$, $\mathbf{H}_1' \boldsymbol{R} \mathbf{H}_2'$ iff either $\mathbf{H}_1' = \mathbf{G}'$ and $\mathbf{H}_2' = \mathbf{G}$ or $\mathbf{G}'$, or $\mathbf{H}_1' \in \mathbf{K}$, $\mathbf{H}_2' \in \mathbf{K}$ and $\mathbf{H}_1' \boldsymbol{R} \mathbf{H}_2'$. Let φ' be an extension of φ to $(\mathbf{G}', \mathbf{K}', \boldsymbol{R}')$; i.e., $\varphi'(P, \mathbf{H}) = \varphi(P, \mathbf{H})$ for $\mathbf{H} \in \mathbf{K}$. Then it is easily proved by induction that $\varphi'(B, \mathbf{H}) = \varphi(B, \mathbf{H})$ for $\mathbf{H} \in \mathbf{K}$ and any subformula B of A. So $\varphi'(A, \mathbf{G}) = \varphi(A, \mathbf{G}) = \mathbf{F}$. Since $\mathbf{G}' \boldsymbol{R} \mathbf{G}$, $\varphi'(\Box A, \mathbf{G}') = \mathbf{F}$. Since $\mathbf{G}' \boldsymbol{R} \mathbf{G}'$, $(\mathbf{G}', \mathbf{K}', \boldsymbol{R}')$ is an S2 m.s., hence φ' is an S2 model, so that $\Box A$ is not valid, hence not provable, in S2. Similarly for $S2^n$ and S3. (In the case of $S2^n$, $n > 1$, instead of adjoining a single element $\mathbf{G}'$ to $\mathbf{K}$, we form $\mathbf{K}'$ by adjoining a string of new elements $\mathbf{G}_1, \ldots, \mathbf{G}_n$, and stipulating that $\mathbf{G}_i \boldsymbol{R}' \mathbf{G}_i$ $(i = 1, \ldots, n)$, $\mathbf{G}_i \boldsymbol{R}' \mathbf{G}_{i+1}$ $(i = 1, \ldots, n-1)$, and $\mathbf{G}_n \boldsymbol{R}' \mathbf{G}$. Further for $\mathbf{H}, \mathbf{H}' \in \mathbf{K}$, $\mathbf{H} \boldsymbol{R}' \mathbf{H}'$ iff $\mathbf{H} \boldsymbol{R} \mathbf{H}'$. Form $(\mathbf{G}_1, \mathbf{K}', \boldsymbol{R}')$, where $\boldsymbol{R}'$ holds only as required above; this is clearly an $S2^n$ m.s. If φ' extends φ to $(\mathbf{G}_1, \mathbf{K}', \boldsymbol{R}')$, then $\varphi(\Box^n A, \mathbf{G}_1) = \mathbf{F}$ if $\varphi(A, \mathbf{G}) = \mathbf{F}$.) Q.E.D.

We can now, incidentally, show that the axiomatizations of S2 and S3 given here are in fact equivalent to the original systems of Lewis (Lewis and Langford [59]). Actually, we prove their equivalence to Lemmon's P2 and P3, respectively; a proof that Lemmon's systems are equivalent to Lewis's forms of S2 and S3 has been given in Lemmon [57].

The *axioms* of P2 (P3) are all axioms of E2 (E3). In addition to modus ponens, Lemmon's rules for P2 (P3) are a rule stipulating that if A is a tautology or an axiom, $\vdash \Box A$; and a rule $\Box(A \supset B)/\Box(\Box A \supset \Box B)$. To derive the first rule in S2 (S3), as it has been formulated here, observe that if A is a tautology or an axiom of P2 (P3), it is a theorem of E2 (E3); so, by Theorem 6, $\Box A$ is a theorem of S2 (S3). For the second rule, observe that if $\vdash \Box (A \supset B)$ in S2 (S3), by Theorem 6 $\vdash A \supset B$ in E2 (E3), hence by (Eb) $\vdash \Box A \supset \Box B$ in E2 (E3), and hence by Theorem 6 $\vdash \Box (\Box A \supset \Box B)$ in S2 (S3). So S2 (S3) contains P2 (P3); the converse inclusion is trivial.

We notice that, as in Kripke [63], the tableaux lead to a decision procedure for the propositional calculi considered. In fact, in $S2^n$, a formula A of degree m either is valid or has a finite tree countermodel, in which each branch of the tree is of length $\leqslant m$. The proof is like that for M in Kripke [63]. In E3 and S3, we prove, similarly to the proof for S4 in Kripke [63], that a formula A either is valid or has both a finite countermodel and a (possibly infinite) denumerable tree countermodel.

Theorem 7. *If A is a formula of degree $\leqslant m$, then $\vdash A$ in $S2^{m+1}$ if and only if $\vdash A$ in* M.

Proof. The "only if" part is trivial. If A is not provable in $S2^{m+1}$, then it has, being of degree $\leqslant m$, a finite tree countermodel φ in which each branch is of length $\leqslant m$, and hence if $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is the associated model structure, every $\mathbf{H} \in \mathbf{K}$ has depth $\leqslant m$. Since $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is an $S2^{m+1}$ model structure, this shows that every $\mathbf{H} \in \mathbf{K}$ is normal, and hence that $(\mathbf{G}, \mathbf{K}, \mathbf{R})$ is an M m.s. But then φ is a countermodel to A in M.

Remark. By use of semantic tableaux and a modified proof, we can improve $S2^{m+1}$ in this result to $S2^m$. The latter is a "best possible" result; for no formula $\Box^m(A \supset A)$ $(m \geqslant 1)$ can be provable in $S2^{m+1}$ (otherwise, by Theorem 6, $\vdash \Box (A \supset A)$ in E2).

Corollary. M *has no axiomatization with only finitely many axiom schemata, and with modus ponens as sole rule of inference.*

Proof. If it had such an axiomatization, let m be the maximum modal degree of the schemata [5]. By Theorem 7 [5], every axiom would be provable in $S2^{m+1}$, and since modus ponens is a rule of $S2^{m+1}$, M would be contained in $S2^{m+1}$. But $\Box^{m+2}(A \supset A)$, though a theorem of M, is not a theorem of $S2^{m+1}$, as shown in the remark just made; so the *reductio* is complete.

This corollary has long been a widespread unproved conjecture.

Finally, notice that, as in Kripke [63], given a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ we can define a *proposition* ϱ as a mapping of elements of $\mathbf{K}$ into $\{\mathbf{T}, \mathbf{F}\}$, or, equivalently, as a subset of $\mathbf{K}$. Given propositions ϱ and σ, define $\varrho \wedge \sigma$, $\sim \varrho$ as in Kripke [63]; $\Box \varrho$ is now defined by $\Box \varrho(\mathbf{H}) = \mathbf{T}$ iff $\mathbf{H}$ is normal and $\varrho(\mathbf{H}') = \mathbf{T}$ for all $\mathbf{H}' \in \mathbf{K}$ such that $\mathbf{H} \boldsymbol{R} \mathbf{H}'$; otherwise $\Box \varrho(\mathbf{H}) = \mathbf{F}$.

As in Kripke [63], we get matrices for $S2^n$, E3, S3 from model structures for these systems. Let $\mathbf{K} = \{\mathbf{G}, \mathbf{H}\}$, and let $1 = \{\mathbf{G}, \mathbf{H}\}$, $2 = \{\mathbf{G}\}$, $3 = \{\mathbf{H}\}$, $4 =$ empty set, taking propositions as subsets of $\mathbf{K}$, as in Kripke

[5] Strictly speaking, a single schema like $\Box (A \supset A)$ can have an arbitrarily large modal degree, depending on the modal degree of A. We make the convention that the degree of a schema is to be computed as if the schematic letters therein are atomic formulae; thus $\Box (A \supset A)$ has degree 1. Note that if a schema has degree m in this sense, we can replace distinct schematic letters by distinct propositional variables to obtain a formula B of degree m in the ordinary sense. Hence, by Theorem 7, $\vdash B$ in $S2^{m+1}$. But since $S2^{m+1}$ contains the rule of substitution as a derived rule, the original schema thus is derivable in $S2^{m+1}$.

[63], p. 93. If **G *R* G**, **G *R* H**, but neither **H *R* G** nor **H *R* H**, (**G**, **K**, ***R***) is an S3 m.s., and the matrix in terms of 1, 2, 3, 4, again with 1 and 2 as designated values, coincides with Group I of Lewis-Langford [59].

Notice that it is impossible to obtain infinite characteristic matrices for $S2^n$, E3, and S3 of the types obtained in Kripke [63] for the systems considered there; for such characteristic matrices would be normal, contrary to Halldén's results.

It should be mentioned that, by a proof like that in Kripke [63], Section 5.3, we can prove the following theorem on S2 and S3: If $\vdash \Box A \vee \Box B$, then $\vdash \Box A$ or $\vdash \Box B$. Similarly in $S2^n$, $n \geqslant 1$, if $\vdash \Box^n A \vee \Box^n B$, then $\vdash \Box^n A$ or $\vdash \Box^n B$. For all the E-systems, the property is trivial: since "$\Box$" is interpretable as a *falsum* operator, no formula $\Box A \vee \Box B$ can be derivable.

7. Other systems. The literature contains systems S6 ($S2 + \Diamond \Diamond A$ [6]), S7 ($= S3 + \Diamond \Diamond A$ [6]), S8 ($= S3 + \Box \Diamond \Diamond A$ [6]). Theorem 2 implies [6] that $S3 = S4 \cap S7$, $S2 = S2^2 \cap S6$, since $\vdash \Diamond \Diamond A \vee (\Box B \supset \Box \Box B)$ in S3 and $\vdash \Diamond \Diamond A \vee \Box \Box (B \supset B)$ in S2. If we define an S6 (S7) m.s. to be an S2 (S3) m.s. (**G**, **K**, ***R***) with at least one $\mathbf{H} \in \mathbf{K}$ of depth 1 which is not normal (*i.e.*, to be an S2 (S3) m.s. which is not an $S2^2$ ($S3^2$) m.s.), and an S8 m.s. to be an S3 m.s. in which for every normal $\mathbf{H} \in \mathbf{K}$ there is an $\mathbf{H}' \in \mathbf{K}$, such that **H *R* H′**, and which is *not* normal, we can derive completeness theorems for the resulting systems.

Finally, if we drop quasi-reflexivity from the requirements on ***R***, and simply define a model structure to be a quadruple (**G**, **K**, ***R***, **N**), where **N** is the set of normal elements, we get a model theory for Lemmon's D2. If ***R*** is required to be transitive, the theory works for D3.

[6] Actually, S6 is usually defined as the result of adding $\Diamond \Diamond A$ to *Lewis's* formulation of S2 (Lewis and Langford [59]), which (unlike our formulation) contains rules of inference other than R1; similarly for S7 and S8. We can show, however, that if $\Diamond \Diamond A$ is added to S2 as formulated above (with R1 as sole rule), Lewis's rules are derivable; and similarly for S7 and S8. Thus there is no problem about applying Theorem 2, which assumes that R1 is the only primitive rule in the systems considered.

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THE FRAENKEL-MOSTOWSKI METHOD FOR INDEPENDENCE PROOFS IN SET THEORY *

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1. Introduction. The Fraenkel–Mostowski (FM) method as introduced by Fraenkel [22, 28, 28a, 32, 35, 37] and Mostowski [38, 38a, 39] turned out to be very useful in proving the independence of the axiom of choice and related statements. Until the recent appearance of Cohen [63, 64] it was the only method available for obtaining such results. The major deficiency of the FM method is that it does not apply to "ordinary" set theories, i.e., to set theories in which all the elements are sets and in which the axiom of foundation holds. The FM method is applied to the following two types of set theories. The first type, the one which Fraenkel and Mostowski originally dealt with, are set theories which admit the existence of urelements, i.e., elements which are not sets. These set theories are obtained from the "ordinary" set theories by what essentially amounts to a weakening of the axiom of extensionality. The second type, to which this method was applied by Mendelson [56], Shoenfield [55] and Specker [57], are set theories which permit the existence of unfounded sets, i.e., they permit the existence of sequences of the type $\ldots \in x_n \in x_{n-1} \in \ldots \in x_2 \in x_1$ or other similar phenomena. These set theories are obtained by weakening the axiom of foundation. The same proofs of independence were obtained in both types of set theories, except for a proof of the independence of the axiom of foundation from some other statements in Mendelson [58]. Since all other known proofs are carried out in, practically, the same way in both types of set theories we shall henceforth mention only the first type, tacitly assuming that everything applies to the second type as well.

The new method of Cohen [63, 64] for proving the independence of the axiom of choice and related statements is stronger than the FM method in that it applies also to "ordinary" set theories. However, independence

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