
PrSAT: Some Examples

First, load the **PrSAT** package:

```
<< PrSAT`
```

■ Example #1: X, Y, Z pairwise independent, but *not independent*

```
MODEL1 =  
PrSAT [  
  {  
    Pr[X ∧ Y] == Pr[X] Pr[Y],  
    Pr[X ∧ Z] == Pr[X] Pr[Z],  
    Pr[Z ∧ Y] == Pr[Z] Pr[Y],  
    Pr[X ∧ Y ∧ Z] ≠ Pr[X] Pr[Y] Pr[Z]  
  },  
  Probabilities → Regular  
]  
  
{ {X → {a2, a5, a6, a8},  
  Y → {a3, a5, a7, a8}, Z → {a4, a6, a7, a8},  
  Ω → {a1, a2, a3, a4, a5, a6, a7, a8}}},  
  
{ a1 →  $\frac{295\,043 - 117\sqrt{4\,676\,097}}{168\,831}$ ,  
  a2 →  $\frac{-84\,085 + 39\sqrt{4\,676\,097}}{168\,831}$ ,  
  a3 →  $\frac{-84\,085 + 39\sqrt{4\,676\,097}}{168\,831}$ ,  
  a4 →  $\frac{-84\,085 + 39\sqrt{4\,676\,097}}{168\,831}$ , a5 →  $\frac{42}{169}$ ,  
  a6 →  $\frac{42}{169}$ , a7 →  $\frac{42}{169}$ , a8 →  $\frac{1}{999}$  } }
```

TruthTable [MODEL1]

X	Y	Z	var	Pr
T	T	T	a_8	$\frac{1}{999}$
T	T	F	a_5	$\frac{42}{169}$
T	F	T	a_6	$\frac{42}{169}$
T	F	F	a_2	$\frac{-84\,085+39\sqrt{4\,676\,097}}{168\,831}$
F	T	T	a_7	$\frac{42}{169}$
F	T	F	a_3	$\frac{-84\,085+39\sqrt{4\,676\,097}}{168\,831}$
F	F	T	a_4	$\frac{-84\,085+39\sqrt{4\,676\,097}}{168\,831}$
F	F	F	a_1	$\frac{295\,043-117\sqrt{4\,676\,097}}{168\,831}$

EvaluateProbability[

{

Pr [X ∧ Y] == Pr [X] Pr [Y] ,

Pr [X ∧ Z] == Pr [X] Pr [Z] ,

Pr [Z ∧ Y] == Pr [Z] Pr [Y] ,

Pr [X ∧ Y ∧ Z] ≠ Pr [X] Pr [Y] Pr [Z]

}, MODEL1]

{True, True, True, True}

■ Example #2: s violates (†)

$s[h_, e_] := \text{Pr}[h | e] - \text{Pr}[h | \neg e];$

```

MODEL2 =
  PrSAT[
    {
      Pr[H | E1] > Pr[H | E2],
      s[H, E1] < s[H, E2]
    },
    Probabilities → Regular]

{ {E1 → {a2, a5, a6, a8},
  E2 → {a3, a5, a7, a8}, H → {a4, a6, a7, a8},
  Ω → {a1, a2, a3, a4, a5, a6, a7, a8}},
  { a1 →  $\frac{510\,368\,209}{3\,344\,528\,124}$ , a2 →  $\frac{2}{31}$ , a3 →  $\frac{4}{21}$ , a4 →  $\frac{1}{999}$ ,
    a5 →  $\frac{1}{49}$ , a6 →  $\frac{1}{532}$ , a7 →  $\frac{10}{37}$ , a8 →  $\frac{26}{87}$  } }

```

TruthTable [MODEL2]

E1	E2	H	var	Pr
T	T	T	a ₈	$\frac{26}{87}$
T	T	F	a ₅	$\frac{1}{49}$
T	F	T	a ₆	$\frac{1}{532}$
T	F	F	a ₂	$\frac{2}{31}$
F	T	T	a ₇	$\frac{10}{37}$
F	T	F	a ₃	$\frac{4}{21}$
F	F	T	a ₄	$\frac{1}{999}$
F	F	F	a ₁	$\frac{510\,368\,209}{3\,344\,528\,124}$

```

EvaluateProbability[
  {
    Pr[H | E1] > Pr[H | E2],
    s[H, E1] < s[H, E2]
  }, MODEL2]

```

{True, True}

Note: d satisfies (\dagger)

```
d[h_, e_] := Pr[h | e] - Pr[h];
```

```
PrSAT[  
  {  
    Pr[H | E1] ≥ Pr[H | E2],  
    d[H, E1] < d[H, E2]  
  }  
]
```

```
– PrSAT::srchfail :  
  Search phase failed; attempting  
  FindInstance  
  
  {}
```

■ Example #3: A Counterexample Requiring 4-Events (Douven)

Question: Given a finite set S of propositions such that (1) each proposition H in S is correlated with the conjunction of the members of any non-empty subset of S that does not contain H , is (2) the conjunction of the members of any non-empty subset S' of S correlated with the conjunction of any non-empty subset S'' of S not overlapping with S' ?

Answer: No. Note: the answer is YES for 3-event spaces. So, we assume a four-event probability space, over the propositions $\{X, Y, Z, W\}$. To try to find a 4-element countermodel, we first represent condition (1):

```

C1 :=
{
  Pr[X | Y] > Pr[X] ,
  Pr[X | Z] > Pr[X] ,
  Pr[X | W] > Pr[X] ,
  Pr[Y | Z] > Pr[Y] ,
  Pr[Y | W] > Pr[Y] ,
  Pr[Z | W] > Pr[Z] ,
  Pr[X | Y ∧ Z] > Pr[X] ,
  Pr[X | Y ∧ W] > Pr[X] ,
  Pr[X | Z ∧ W] > Pr[X] ,
  Pr[Y | X ∧ Z] > Pr[Y] ,
  Pr[Y | X ∧ W] > Pr[Y] ,
  Pr[Y | Z ∧ W] > Pr[Y] ,
  Pr[Z | X ∧ Y] > Pr[Z] ,
  Pr[Z | X ∧ W] > Pr[Z] ,
  Pr[Z | Y ∧ W] > Pr[Z] ,
  Pr[X | Y ∧ Z ∧ W] > Pr[X] ,
  Pr[Y | X ∧ Z ∧ W] > Pr[Y] ,
  Pr[Z | X ∧ Y ∧ W] > Pr[Z]
};

```

Now, the key will be trying to find models in which **C1** (*i.e.*, all the inequalities in **C1**) holds, and in which at least one of the following equalities holds (these would, of course, be models in which (2) *fails*):

```

C2 :=
{
  Pr[X ∧ Y | Z ∧ W] == Pr[X ∧ Y] ,
  Pr[X ∧ Z | Y ∧ W] == Pr[X ∧ Z] ,
  Pr[Y ∧ Z | X ∧ W] == Pr[Y ∧ Z]
}

```

In fact, we'll find a model in which **all** the equations in **C2** hold, using our **PrSAT** function, as follows:

MODEL3 =

PrSAT [

Union[C1, C2],

Probabilities → Regular

]

$$\left\{ \begin{array}{l} W \rightarrow \{a_2, a_6, a_7, a_8, a_{12}, a_{13}, a_{14}, a_{16}\}, \\ X \rightarrow \{a_3, a_6, a_9, a_{10}, a_{12}, a_{13}, a_{15}, a_{16}\}, \\ Y \rightarrow \{a_4, a_7, a_9, a_{11}, a_{12}, a_{14}, a_{15}, a_{16}\}, \\ Z \rightarrow \{a_5, a_8, a_{10}, a_{11}, a_{13}, a_{14}, a_{15}, a_{16}\}, \\ \Omega \rightarrow \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \\ a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}\} \end{array} \right\},$$
$$\left\{ \begin{array}{l} a_1 \rightarrow \frac{53\,930\,355\,133\,083\,251\,854}{389\,885\,438\,824\,209\,942\,075}, \quad a_2 \rightarrow \frac{2}{45}, \\ a_3 \rightarrow \frac{2}{45}, \quad a_4 \rightarrow \frac{1}{25}, \quad a_5 \rightarrow \frac{2}{45}, \quad a_6 \rightarrow \frac{91\,386\,781}{2\,219\,960\,050}, \\ a_7 \rightarrow \frac{372\,759}{8\,066\,450}, \quad a_8 \rightarrow \frac{13\,109\,319}{290\,102\,450}, \\ a_9 \rightarrow \frac{1}{18}, \quad a_{10} \rightarrow \frac{3}{55}, \quad a_{11} \rightarrow \frac{2}{33}, \quad a_{12} \rightarrow \frac{1}{14}, \\ a_{13} \rightarrow \frac{4}{55}, \quad a_{14} \rightarrow \frac{3}{38}, \quad a_{15} \rightarrow \frac{5}{61}, \quad a_{16} \rightarrow \frac{2}{25} \end{array} \right\}$$

TruthTable [MODEL3]

W	X	Y	Z	var	Pr
T	T	T	T	a ₁₆	$\frac{2}{25}$
T	T	T	F	a ₁₂	$\frac{1}{14}$
T	T	F	T	a ₁₃	$\frac{4}{55}$
T	T	F	F	a ₆	$\frac{91\,386\,781}{2\,219\,960\,050}$
T	F	T	T	a ₁₄	$\frac{3}{38}$
T	F	T	F	a ₇	$\frac{372\,759}{8\,066\,450}$
T	F	F	T	a ₈	$\frac{13\,109\,319}{290\,102\,450}$
T	F	F	F	a ₂	$\frac{2}{45}$
F	T	T	T	a ₁₅	$\frac{5}{61}$
F	T	T	F	a ₉	$\frac{1}{18}$
F	T	F	T	a ₁₀	$\frac{3}{55}$
F	T	F	F	a ₃	$\frac{2}{45}$
F	F	T	T	a ₁₁	$\frac{2}{33}$
F	F	T	F	a ₄	$\frac{1}{25}$
F	F	F	T	a ₅	$\frac{2}{45}$
F	F	F	F	a ₁	$\frac{53\,930\,355\,133\,083\,251\,854}{389\,885\,438\,824\,209\,942\,075}$

EvaluateProbability[

```
{  
  Union[C1, C2]  
}, MODEL3]
```

```
{{True, True, True, True, True, True, True,  
  True, True, True, True, True, True, True,  
  True, True, True, True, True, True, True}}
```

■ "Proving" Theorems

Bayes's Theorem is trivial:

$$\text{PrSAT} \left[\begin{array}{l} \{ \\ \text{Pr}[X | Y] \neq \\ \frac{\text{Pr}[Y | X] \text{Pr}[X]}{\text{Pr}[Y | X] \text{Pr}[X] + \text{Pr}[Y | \neg X] \text{Pr}[\neg X]} \\ \} \\ \end{array} \right]$$

- PrSAT::srchfail :
 Search phase failed; attempting
 FindInstance
 { }

Popper-Miller Theorem "Additivity of *d*-Confirmation" Theorem is trivial:

$$\text{PrSAT} \left[\begin{array}{l} \{ \\ d[H, E] \neq d[H \vee E, E] + d[H \vee \neg E, E] \\ \} \\ \end{array} \right]$$

- PrSAT::srchfail :
 Search phase failed; attempting
 FindInstance
 { }

```
PrSAT [
  {
    d[H ∨ ¬ E, E] > 0
  }
]
```

- PrSAT::srchfail :
Search phase failed; attempting
FindInstance

```
{}
```

```
PrSAT [
  {
    d[H, E] ≠ d[H ∧ E, E] + d[H ∧ ¬ E, E]
  }
]
```

- PrSAT::srchfail :
Search phase failed; attempting
FindInstance

```
{}
```

```
PrSAT [
  {
    d[H ∧ ¬ E, E] > 0
  }
]
```

- PrSAT::srchfail :
Search phase failed; attempting
FindInstance

```
{}
```

Note: Popper-Miller Theorem *fails* for r (first pointed out by Redhead)

```
r[h_, e_] := Pr[h | e] / Pr[h];
```

MODEL4 = PrSAT [
 {
 r[H, E] ≠ r[H ∨ E, E] + r[H ∨ ¬ E, E]
 },
 Probabilities → Regular
]

{ {E → {a₂, a₄},
 H → {a₃, a₄}, Ω → {a₁, a₂, a₃, a₄}},
 {a₁ → $\frac{427}{285\,714}$, a₂ → $\frac{1}{999}$, a₃ → $\frac{1}{999}$, a₄ → $\frac{285}{286}$ } }

TruthTable [MODEL4]

E	H	var	Pr
T	T	a ₄	$\frac{285}{286}$
T	F	a ₂	$\frac{1}{999}$
F	T	a ₃	$\frac{1}{999}$
F	F	a ₁	$\frac{427}{285\,714}$

Wagner's "Probabilistic Modus Tollens" Theorem is trivial

$$\text{PrSAT} \left[\begin{array}{l} \left\{ \begin{array}{l} \frac{1 - \text{Pr}[\text{E} \mid \text{H}] - \text{Pr}[\neg \text{E}]}{1 - \text{Pr}[\text{E} \mid \text{H}]} \geq \frac{\text{Pr}[\text{E} \mid \text{H}] + \text{Pr}[\neg \text{E}] - 1}{\text{Pr}[\text{E} \mid \text{H}]} , \\ \frac{1 - \text{Pr}[\text{E} \mid \text{H}] - \text{Pr}[\neg \text{E}]}{1 - \text{Pr}[\text{E} \mid \text{H}]} > \text{Pr}[\neg \text{H}] \end{array} \right. \\ \left. \right\} \end{array} \right]$$

– PrSAT::srchfail :
 Search phase failed; attempting
 FindInstance
 {}

$$\text{PrSAT} \left[\begin{array}{l} \left\{ \begin{array}{l} \frac{\text{Pr}[\text{E} \mid \text{H}] + \text{Pr}[\neg \text{E}] - 1}{\text{Pr}[\text{E} \mid \text{H}]} \geq \frac{1 - \text{Pr}[\text{E} \mid \text{H}] - \text{Pr}[\neg \text{E}]}{1 - \text{Pr}[\text{E} \mid \text{H}]} , \\ \frac{\text{Pr}[\text{E} \mid \text{H}] + \text{Pr}[\neg \text{E}] - 1}{\text{Pr}[\text{E} \mid \text{H}]} > \text{Pr}[\neg \text{H}] \end{array} \right. \\ \left. \right\} \end{array} \right]$$

– PrSAT::srchfail :
 Search phase failed; attempting
 FindInstance
 {}