

A Better Bayesian Convergence Theorem

Notation

hypotheses: h_i h_j ...

background and auxiliaries: b

experimental/observation conditions:

$c_1, c_2, \dots, c_n : c^n$

evidential outcomes:

$e_1, e_2, \dots, e_n : e^n$

likelihoods: $P[e^n | h_i \cdot b \cdot c^n]$

Notation

Likelihoods

Priors

Posteriors

$$P[e | h_i] \quad : \quad P_\alpha[h_i] \quad \Rightarrow \quad P_\alpha[h_i | e]$$

$$P[e | h_i \cdot b \cdot c] : P_\alpha[h_i | b] \Rightarrow P_\alpha[h_i | b \cdot c \cdot e]$$

$$P_{\alpha}[h_i | b \cdot c^n \cdot e^n]$$

$$= \frac{P[e^n | h_i \cdot b \cdot c^n] \cdot P_{\alpha}[h_i | b]}{P_{\alpha}[e^n | b \cdot c^n]} \cdot \frac{P_{\alpha}[c^n | h_i \cdot b]}{P_{\alpha}[c^n | b]}$$

$$= \frac{P[e^n | h_i \cdot b \cdot c^n] \cdot P_{\alpha}[h_i | b]}{P_{\alpha}[e^n | b \cdot c^n]}$$

$$\frac{P_{\alpha}[h_j | \mathbf{b} \cdot \mathbf{c}^n \cdot \mathbf{e}^n]}{P_{\alpha}[h_i | \mathbf{b} \cdot \mathbf{c}^n \cdot \mathbf{e}^n]}$$

$$= \frac{P[e^n | h_j \cdot \mathbf{b} \cdot \mathbf{c}^n]}{P[e^n | h_i \cdot \mathbf{b} \cdot \mathbf{c}^n]} \cdot \frac{P_{\alpha}[h_j | \mathbf{b}]}{P_{\alpha}[h_i | \mathbf{b}]} \cdot \frac{P_{\alpha}[\mathbf{c}^n | h_j \cdot \mathbf{b}]}{P_{\alpha}[\mathbf{c}^n | h_i \cdot \mathbf{b}]}$$

$$= \frac{P[e^n | h_j \cdot \mathbf{b} \cdot \mathbf{c}^n]}{P[e^n | h_i \cdot \mathbf{b} \cdot \mathbf{c}^n]} \cdot \frac{P_{\alpha}[h_j | \mathbf{b}]}{P_{\alpha}[h_i | \mathbf{b}]}$$

$$= \frac{P[e^n | h_j \cdot \mathbf{b} \cdot \mathbf{c}^n]}{P[e^n | h_i \cdot \mathbf{b} \cdot \mathbf{c}^n]} \cdot \frac{P_{\alpha}[h_j | \mathbf{b}]}{P_{\alpha}[h_i | \mathbf{b}]}$$

$$\begin{aligned}
\Omega_{\alpha}[\sim h_i \mid \mathbf{b} \cdot \mathbf{c}^n \cdot \mathbf{e}^n] &= \frac{P_{\alpha}[\sim h_i \mid \mathbf{b} \cdot \mathbf{c}^n \cdot \mathbf{e}^n]}{P_{\alpha}[h_i \mid \mathbf{b} \cdot \mathbf{c}^n \cdot \mathbf{e}^n]} \\
&= \sum_{j \neq i} \frac{P[e^n \mid h_j \cdot \mathbf{b} \cdot \mathbf{c}^n]}{P[e^n \mid h_i \cdot \mathbf{b} \cdot \mathbf{c}^n]} \cdot \frac{P_{\alpha}[h_j \mid \mathbf{b}]}{P_{\alpha}[h_i \mid \mathbf{b}]} \\
&\quad + \frac{P_{\alpha}[e^n \mid h_K \cdot \mathbf{b} \cdot \mathbf{c}^n]}{P[e^n \mid h_i \cdot \mathbf{b} \cdot \mathbf{c}^n]} \cdot \frac{P_{\alpha}[h_K \mid \mathbf{b}]}{P_{\alpha}[h_i \mid \mathbf{b}]}
\end{aligned}$$

where h_K is the catch-all, “something-else” hypothesis.

$$P_{\alpha}[h_i | \mathbf{b} \cdot \mathbf{c}^n \cdot \mathbf{e}^n] =$$

1

$$1 + \Omega_{\alpha}[\sim h_i | \mathbf{b} \cdot \mathbf{c}^n \cdot \mathbf{e}^n]$$

$$\sum_{j \neq i} \frac{P[e^n | h_j \cdot b \cdot c^n]}{P[e^n | h_i \cdot b \cdot c^n]} \cdot \frac{P_\alpha[h_j | b]}{P_\alpha[h_i | b]}$$

$$\leq \Omega_\alpha[\sim h_i | b \cdot c^n \cdot e^n] \leq$$

$$\sum_{j \neq i} \frac{P[e^n | h_j \cdot b \cdot c^n]}{P[e^n | h_i \cdot b \cdot c^n]} \cdot \frac{P_\alpha[h_j | b]}{P_\alpha[h_i | b]}$$

$$+ \frac{1}{P[e^n | h_i \cdot b \cdot c^n]} \cdot \frac{P_\alpha[h_K | b]}{P_\alpha[h_i | b]}$$

$$\frac{P_{\alpha}[h_j | \mathbf{b} \cdot \mathbf{c}^n \cdot \mathbf{e}^n]}{\frac{P_{\alpha}[h_i | \mathbf{b} \cdot \mathbf{c}^n \cdot \mathbf{e}^n]}{P[e^n | h_i \cdot \mathbf{b} \cdot \mathbf{c}^n]} \cdot \frac{P_{\alpha}[h_j | \mathbf{b}]}{P_{\alpha}[h_i | \mathbf{b}]}}$$

Sufficient Conditions for the
Likely
Bayesian Refutation
of False Alternatives to the
True Hypothesis

choose any $\varepsilon > 0$

consider the set of outcome streams:

$$\{e^n : P[e^n | h_j \cdot b \cdot c^n] / P[e^n | h_i \cdot b \cdot c^n] < \varepsilon\}$$

now consider the sentence:

$$\forall \{e^n : P[e^n | h_j \cdot b \cdot c^n] / P[e^n | h_i \cdot b \cdot c^n] < \varepsilon\}$$

Consider

$$P[\bigvee \{e^n : P[e^n | h_j \cdot b \cdot c^n] / P[e^n | h_i \cdot b \cdot c^n] < \varepsilon\} | h_i \cdot b \cdot c^n]$$

The Outcome Space

c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
$O_{1,1}$	$O_{2,1}$	$O_{3,1}$	$O_{4,1}$	$O_{5,1}$	$O_{6,1}$	$O_{7,1}$	$O_{8,1}$...
$O_{1,2}$	$O_{2,2}$	$O_{3,2}$	$O_{4,2}$	$O_{5,2}$	$O_{6,2}$	$O_{7,2}$	$O_{8,2}$...
$O_{1,3}$	$O_{2,3}$	$O_{3,3}$	$O_{4,3}$	$O_{5,3}$	$O_{6,3}$	$O_{7,3}$	$O_{8,3}$...
$O_{1,4}$	$O_{2,4}$	$O_{3,4}$	$O_{4,4}$	$O_{5,4}$	$O_{6,4}$	$O_{7,4}$	$O_{8,4}$...
$O_{1,5}$	$O_{2,5}$	$O_{3,5}$	$O_{4,5}$	$O_{5,5}$		$O_{7,5}$	$O_{8,5}$...
$O_{1,6}$	$O_{2,6}$	$O_{3,6}$	$O_{4,6}$	$O_{5,6}$		$O_{7,6}$	$O_{8,6}$...
$O_{1,7}$		$O_{3,7}$	$O_{4,7}$	$O_{5,7}$		$O_{7,7}$	$O_{8,7}$...
$O_{1,8}$			$O_{4,8}$	$O_{5,8}$		$O_{7,8}$	$O_{8,8}$...
...		

for each h ,

$$P[o_{ku} \cdot o_{kv} \mid h \cdot b \cdot c_k] = 0$$

$$\sum_{u=1}^w P[o_{ku} \mid h \cdot b \cdot c_k] = 1$$

Possible Path of Evidence Stream through the Outcome Space

e^n

		C								
		c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
e^n	$O_{1,1}$	$O_{1,1}$	$O_{2,1}$	$O_{3,1}$	$O_{4,1}$	$O_{5,1}$	$O_{6,1}$	$O_{7,1}$	$O_{8,1}$...
	$O_{1,2}$	$O_{1,2}$	$O_{2,2}$	$O_{3,2}$	$O_{4,2}$	$O_{5,2}$	$O_{6,2}$	$O_{7,2}$	$O_{8,2}$...
	$O_{1,3}$	$O_{1,3}$	$O_{2,3}$	$O_{3,3}$	$O_{4,3}$	$O_{5,3}$	$O_{6,3}$	$O_{7,3}$	$O_{8,3}$...
	$O_{1,4}$	$O_{1,4}$	$O_{2,4}$	$O_{3,4}$	$O_{4,4}$	$O_{5,4}$	$O_{6,4}$	$O_{7,4}$	$O_{8,4}$...
	$O_{1,5}$	$O_{1,5}$	$O_{2,5}$	$O_{3,5}$	$O_{4,5}$	$O_{5,5}$		$O_{7,5}$	$O_{8,5}$...
	$O_{1,6}$	$O_{1,6}$	$O_{2,6}$	$O_{3,6}$	$O_{4,6}$	$O_{5,6}$		$O_{7,6}$	$O_{8,6}$...
	$O_{1,7}$	$O_{1,7}$		$O_{3,7}$	$O_{4,7}$	$O_{5,7}$		$O_{7,7}$	$O_{8,7}$...
	$O_{1,8}$	$O_{1,8}$			$O_{4,8}$	$O_{5,8}$		$O_{7,8}$	$O_{8,8}$...

Possible Path of
Evidence
Stream
through the
Outcome Space

e^n

$h_i \cdot b$

		C								
		c₁	c₂	c₃	c₄	c₅	c₆	c₇	c₈	c₉
e^n	$h_1 \cdot b$	O _{1,1}	O _{2,1}	O _{3,1}	O _{4,1}	O _{5,1}	O _{6,1}	O _{7,1}	O _{8,1}	...
	$h_2 \cdot b$	O _{1,2}	O _{2,2}	O _{3,2}	O _{4,2}	O _{5,2}	O _{6,2}	O _{7,2}	O _{8,2}	...
	$h_3 \cdot b$	O _{1,3}	O _{2,3}	O _{3,3}	O _{4,3}	O _{5,3}	O _{6,3}	O _{7,3}	O _{8,3}	...
	$h_4 \cdot b$	O _{1,4}	O _{2,4}	O _{3,4}	O _{4,4}	O _{5,4}	O _{6,4}	O _{7,4}	O _{8,4}	...
	$h_5 \cdot b$	O _{1,5}	O _{2,5}	O _{3,5}	O _{4,5}	O _{5,5}		O _{7,5}	O _{8,5}	...
	$h_6 \cdot b$	O _{1,6}	O _{2,6}	O _{3,6}	O _{4,6}	O _{5,6}		O _{7,6}	O _{8,6}	...
	$h_7 \cdot b$	O _{1,7}		O _{3,7}	O _{4,7}	O _{5,7}		O _{7,7}	O _{8,7}	...
	$h_8 \cdot b$	O _{1,8}			O _{4,8}	O _{5,8}		O _{7,8}	O _{8,8}	...

$$P[\bigvee \{e^n : P[e^n \mid h_j \cdot b \cdot c^n] / P[e^n \mid h_i \cdot b \cdot c^n] < \varepsilon\} \mid h_i \cdot b \cdot c^n]$$

Independent Evidence Assumptions:

$$1. P[e^k | h_j \cdot b \cdot c_{k+1} \cdot c^k] = P[e^k | h_j \cdot b \cdot c^k];$$

$$2. P[e_{k+1} | h_j \cdot b \cdot c_{k+1} \cdot c^k \cdot e^k] = P[e_{k+1} | h_j \cdot b \cdot c_{k+1}]$$

$$\therefore P[e^n | h_j \cdot b \cdot c^n] = \prod_{k=1}^n P[e_k | h_j \cdot b \cdot c_k]$$

Definition: o_{ku} is a **falsifying outcome** of c_k for h_j with respect to h_i iff

$$P[o_{ku} \mid h_j \cdot b \cdot c_k] = 0 \text{ but } P[o_{ku} \mid h_i \cdot b \cdot c_k] > 0$$

Definition: h_j is **outcome-compatible** with h_i on c_k iff none of the outcomes of c_k are **falsifying** for h_j with respect to h_i

Theorem 1: The Falsification Theorem:

Suppose c^n contains a sub-sequence consisting of m experiments or observations such that for each of them the likelihood of obtaining a *falsifying outcome* is no less than some number $\delta > 0$

$$\text{i.e., } P[\bigvee \{o_{ku} : P[o_{ku} | h_j \cdot b \cdot c_k] = 0\} | h_i \cdot b \cdot c_k] \geq \delta.$$

Then,

$$P[\bigvee \{e^n : P[e^n | h_j \cdot b \cdot c^n] / P[e^n | h_i \cdot b \cdot c^n] = 0\} | h_i \cdot b \cdot c^n] \\ \geq 1 - (1 - \delta)^m.$$

(Notice: if there is a *crucial experiment* in evidence stream c^n , then we may choose $m = 1$ and $\delta = 1$.)

A measure of the Empirical Distinctness of Hypotheses when Outcome-Compatible on the experiment

Definitions: Quality of Information from an outcome

$$\begin{aligned} \text{QI}[o_{ku} | h_i/h_j | b \cdot c_k] &= \log(P[o_{ku} | h_i \cdot b \cdot c_k] / P[o_{ku} | h_j \cdot b \cdot c_k]) \\ &= \log(P[o_{ku} | h_i \cdot b \cdot c_k]) - \log(P[o_{ku} | h_j \cdot b \cdot c_k]) \end{aligned}$$

$$\text{QI}[e^n | h_i/h_j | b \cdot c^n] = \log(P[e^n | h_i \cdot b \cdot c^n] / P[e^n | h_j \cdot b \cdot c^n])$$

$$\therefore \text{QI}[e^n | h_i/h_j | b \cdot c^n] = \sum_{k=1}^n \text{QI}[e_k | h_i/h_j | b \cdot c_k]$$

Definition: Expected Quality of information for an
observation or experiment:

for c_k on which h_j is outcome-compatible with h_i ,

$$EQI[c_k | h_i/h_j | h_i \cdot b] = \sum_u QI[o_{ku} | h_i/h_j | b \cdot c_k] \cdot P[o_{ku} | h_i \cdot b \cdot c_k]$$

$$EQI[c^n | h_i/h_j | h_i \cdot b] = \sum_{e^n} QI[e^n | h_i/h_j | b \cdot c^n] \cdot P[e^n | h_i \cdot b \cdot c^n]$$

$$\therefore EQI[c^n | h_j/h_i | h_i \cdot b] = \sum_{k=1}^n EQI[c_k | h_j/h_i | h_i \cdot b]$$

Definition: $\underline{EQI}[c^n | h_i/h_j | h_i \cdot b] = EQI[c^n | h_i/h_j | h_i \cdot b] \div n$

Theorem: Boundedness of EQI

$$\text{EQI}[c_k \mid h_j/h_i \mid h_i \cdot b] \geq 0; \text{ and}$$

$$\text{EQI}[c_k \mid h_j/h_i \mid h_i \cdot b] > 0$$

if and only if

for at least one of its possible outcomes o_{ku} ,

$$P[o_{ku} \mid h_i \cdot b \cdot c_k] \neq P[o_{ku} \mid h_j \cdot b \cdot c_k].$$

Definition: Variance in the Quality of Information for c_k :

for c_k on which h_j is outcome-compatible with h_i ,

$$VQI[c_k | h_i/h_j | h_i \cdot b] =$$

$$\sum_{u} (QI[o_{ku} | h_i/h_j | b \cdot c_k] - EQI[c_k | h_i/h_j | h_i \cdot b])^2 \cdot P[o_{ku} | h_i \cdot b \cdot c_k]$$

$$VQI[c^n | h_i/h_j | h_i \cdot b] =$$

$$\sum_{e^n} (QI[e^n | h_i/h_j | b \cdot c^n] - EQI[c^n | h_i/h_j | h_i \cdot b])^2 \cdot P[e^n | h_i \cdot b \cdot c^n]$$

$$\therefore VQI[c^n | h_i/h_j | h_i \cdot b] = \sum_{k=1}^n VQI[c_k | h_i/h_j | h_i \cdot b]$$

Definition: $\underline{VQI}[c^n | h_i/h_j | h_i \cdot b] = VQI[c^n | h_i/h_j | h_i \cdot b] \div n$

Theorem 2:

Non-falsifying Likelihood Ratio Convergence Theorem

Choose positive $\varepsilon < 1$, as small as you like, but large enough that (for the number of observations n being contemplated) the value of $\underline{\text{EQI}}[\mathbf{c}^n \mid \mathbf{h}_i/\mathbf{h}_j \mid \mathbf{h}_i \cdot \mathbf{b}] > -(\log \varepsilon)/n$. Then

$$P[\forall \{e^n : P[e^n \mid \mathbf{h}_j \cdot \mathbf{b} \cdot \mathbf{c}^n]/P[e^n \mid \mathbf{h}_i \cdot \mathbf{b} \cdot \mathbf{c}^n] < \varepsilon\} \mid \mathbf{h}_i \cdot \mathbf{b} \cdot \mathbf{c}^n] \geq$$

$$1 - \frac{1}{n} \cdot \frac{\underline{\text{VQI}}[\mathbf{c}^n \mid \mathbf{h}_i/\mathbf{h}_j \mid \mathbf{h}_i \cdot \mathbf{b}]}{[\underline{\text{EQI}}[\mathbf{c}^n \mid \mathbf{h}_i/\mathbf{h}_j \mid \mathbf{h}_i \cdot \mathbf{b}] + (\log \varepsilon)/n]^2}$$

Theorem 2*:

Non-falsifying Likelihood Ratio Convergence Theorem

Suppose there is some fraction γ , $0 < \gamma \leq (1/e)^2 (\approx .135)$ such that for each possible outcome o_{ku} of each observation condition c_k in c^n , either $P[o_{ku} | h_i \cdot b \cdot c_k] = 0$ or $P[o_{ku} | h_j \cdot b \cdot c_k] / P[o_{ku} | h_i \cdot b \cdot c_k] \geq \gamma$.

Choose positive $\varepsilon < 1$ such that $\underline{EQI}[c^n | h_i/h_j | h_i \cdot b] > -(\log \varepsilon)/n$. Then

$$P[\bigvee \{e^n : P[e^n | h_j \cdot b \cdot c^n] / P[e^n | h_i \cdot b \cdot c^n] < \varepsilon\} | h_i \cdot b \cdot c^n] \geq$$

$$1 - \frac{1}{n} \cdot \frac{(\log \gamma)^2}{[\underline{EQI}[c^n | h_i/h_j | h_i \cdot b] + (\log \varepsilon)/n]^2}$$

Directional Agreement Condition: For each experiment or observation c and each of its possible outcomes o , the *likelihood ratios agree in direction*: i.e.,

$$P_{\alpha}[o | h_j \cdot b \cdot c] / P_{\alpha}[o | h_i \cdot b \cdot c] > 1 \quad \text{iff}$$

$$P_{\beta}[o | h_j \cdot b \cdot c] / P_{\beta}[o | h_i \cdot b \cdot c] > 1, \quad \text{and}$$

$$P_{\alpha}[o | h_j \cdot b \cdot c] / P_{\alpha}[o | h_i \cdot b \cdot c] < 1 \quad \text{iff}$$

$$P_{\beta}[o | h_j \cdot b \cdot c] / P_{\beta}[o | h_i \cdot b \cdot c] < 1.$$