Coherent Choice Functions Under Uncertainty*

Teddy Seidenfeld – joint work with Jay Kadane and Mark Schervish

Carnegie Mellon University

OUTLINE

1. **Preliminaries**
   a. **Coherent choice functions**
   b. **The framework of horse-lotteries and the Anscombe-Aumann theory.**
   c. **Strict preference ($\prec$) *versus* weak preference ($\preceq$) without ordering.**

2. **Some limitations using binary comparisons – and how to avoid them.**

3. **A representation of coherent choice functions – work in progress!**

Part 1: Some preliminaries on coherent choice functions.

The main goal of this investigation is to characterize *coherent* choice functions and to understand the extent to which such *coherence* cannot be reduced to pairwise comparisons between options.

Here, a choice function is called <u>*coherent*</u> if it admits a representation in terms maximizing expected utility with respect to some probability/utility pair $<p, u>$ from a set S of such pairs.

For this discussion, the following two restrictions will be in place:

- **Act-state independence: no cases of "moral hazards" are considered.**
- **State-independent utility: no cases where the value of a prize depends upon the state in which it is received.**

A coherent choice function identifies the Bayes-solutions to a decision problem – the Bayes solutions taken with respect to the set *S*.

That is, given any (closed) set *O* of options, a choice function *C* identifies the admissible options $C[O] = B$, for a non-empty subset $B \subseteq O$.

***Definition*: The choice function is *coherent* with respect to a set *S* of prob/util pairs if:**

for each $b \in B$ there is a pair $\langle p, u \rangle$ in *S* such that *b* maximizes the *p*-expected *u*-utility of options from *O*,

and

***B* constitutes the set of all such solutions.**

B is the set of Bayes-solutions from O: solutions that are Bayes with respect to S. Alternatively, we can examine the *rejection* function $R[O] = O \setminus C[O]$, which identifies the *inadmissible* options in O.

When C is coherent, the inadmissible options are those that fail to maximize p-expected u-utility for <u>all</u> pairs $\langle p, u \rangle$ in S: there is unanimity over S in the rejection of inadmissible options.

***Theorem* (see the Addendum): If an option fails to be admissible in this sense with respect to the class of all probabilities, i.e., when it fails to have a Bayes model, then it is uniformly, strictly dominated by a finite mixture from O.**

In this sense, incoherent choices suffer deFinetti's penalty – being uniformly strictly dominated by a mixed option – within the decision at hand.

Here are three important cases, in increasing order of generality:

- ***S* is a singleton pair, we have traditional *Subjective Expected Utility* theory.**

- ***S* is of the form $P \otimes u$ for a single utility *u* and a closed, convex set of probabilities *P*, and when all option sets are closed under mixtures, we have Walley-Sen's principle of *Maximality*.**

- **When *S* is a cross product of two convex sets $P \otimes U$, we have I. Levi's rule of *E-admissibility*.**

There are several themes that lead to a set *S* that is not a singleton pair.

Here are 3 examples.

1. **Robust Bayesian Analysis – where *S* reflects a (typically) convex set of distributions obtained by varying either/or the *prior* or the *statistical model*, together with a single loss function.**

For example, the *ε-Contamination Model* is obtained by using a set of distns:

$\{(1-\varepsilon)\mu + \varepsilon q$: with μ a fixed distn, and $q \in Q$, where Q is a set of distributions$\}$

The idea is that with probability 1-ε the intended model μ generates the data. And with probability ε any distribution from *Q* might generate the data.

2. **Lower Probability with a (closed) convex set of distributions *P* and a single *u*.**
   - **CAB Smith, P. Walley (and many others), where *S* is based on one-sided previsions.**
   - ***Γ-Maximin* applied to convex sets of bets: Gilboa-Schmeidler theory.**
   - **Dempster-Shafer theory, with a decision rule suggested by Dempster.**

3. **Consensus in cooperative group decision making:**
   - **Levi – where $S = P \otimes U$. *P* is the convex set representing the group's uncertain degrees of belief, and *U* the convex set of its uncertain values.**
   - **SSK – *S* is a set of prob/util pairs that represent the unrestricted Pareto (strict) preferences in a group of coherent decision makers.**

In this presentation, the framework of Anscombe-Aumann horse-lotteries is convenient for focusing on coherent choice functions generated by uncertainty in the decision maker's degrees of belief, while utility is left determinate. For this reason, the choice functions involved in this presentation reflect only *uncertainty*, as there is no indeterminacy or imprecision in the decision maker(s) utilities.

We will use *L* the set of von Neumann-Morgenstern lotteries on two prizes,

b ("better") and w ("worse"),

which serve respectively as the 1 and 0 of the cardinal utility function.

Thus, a von Neumann-Morgenstern lottery *l* is given as the mixture

$$\alpha b \oplus (1-\alpha)w,$$

with determinate utility α.

The fixed finite partition $\Omega = \{\omega_1, ..., \omega_n\}$ is the space of the agent's uncertainty.

An option, a horse-lottery h is a function from Ω to L, from states to lotteries.

	ω_1	ω_2	ω_3	…	ω_n
h	l_1	l_2	l_3	…	l_n

The mixture of two horse lotteries $h_3 = \beta h_1 \oplus (1\text{-}\beta)h_2$ is merely the state-by-state β-mixture of their respective lotteries

	ω_1	ω_2	…	ω_n
h_1	l_{11}	l_{12}	…	l_{1n}
h_2	l_{21}	l_{22}	…	l_{2n}
h_3	$\beta l_{11} \oplus (1\text{-}\beta) l_{21}$	$\beta l_{12} \oplus (1\text{-}\beta) l_{22}$	…	$\beta l_{1n} \oplus (1\text{-}\beta) l_{2n}$

- **The final preliminary point is to understand the importance of using a *strict preference* ($<$) relation, rather than a *weak preference* ($\leq$) relation when dealing with uncertainty represented by coherent choice functions.**

Here is a simple example using a two-state partition, $\Omega = \{\omega_1, \omega_2\}$, corresponding to the outcome of a toss of a coin landing *tails* (ω_1) or *heads* (ω_2).

Let $h_{.5}$ be the constant horse lottery that pays off $\alpha = .5$ in either state.

Let h be the horse lottery that pays w if the *tails* and b if *heads*.

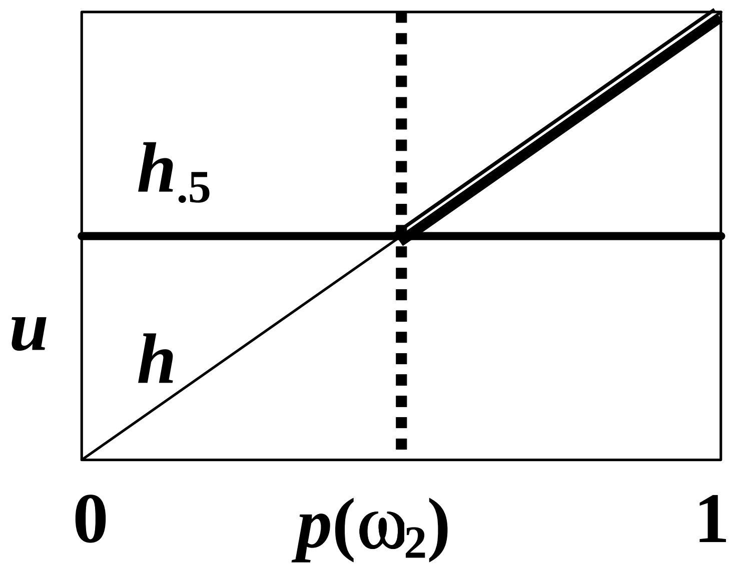
Contrast these two states of uncertainty.

$S_1 = \{p\text{: } p(\omega_1) < p(\omega_2)$**; the coin is biased for *heads*}**

$S_2 = \{p\text{: } p(\omega_1) \leq p(\omega_2)$**; the coin is *not* biased for *tails*}**

In a binary choice from the pair of option $O = \{h, h_{.5}\}$

- **under S_1 only option h maximizes expected utility**
- **under S_2 both options are coherently admissible.**



Strict preference ($\prec$) but not *weak preference* ($\preceq$) captures this distinction (SSK 95).

2. Limitations using strict preference: a binary comparison between acts.

2.1 Distinct (convex) sets of distributions that generate the same strict partial order over horse lotteries.

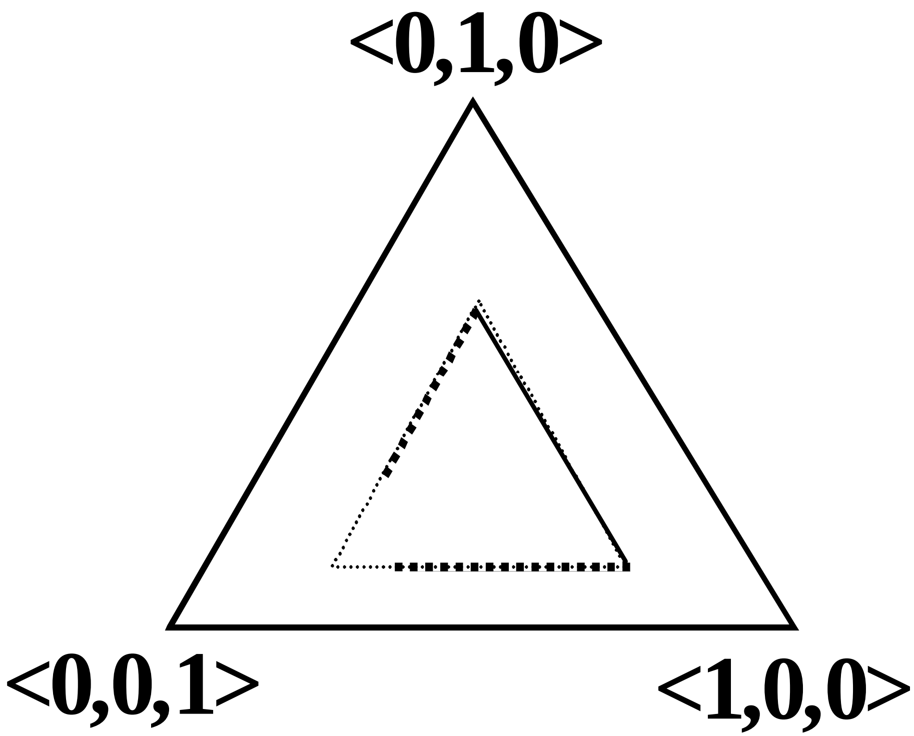
The general problem is that a strict preference between two of our horse lotteries

$$h_1 < h_2$$

defines a hyperplane of separation between the (convex) set of those distributions that give h_2 greater expected utility than h_1, and those distributions that do not.

But many distinct (convex) sets of distributions all may agree on these binary comparisons. They differ with regard to their boundaries.

- **Here is an example using three states: $\Omega = \{\omega_1, \omega_2, \omega_3\}$.**



2.2 A related problem is that choice functions that use binary comparisons between options to determine what is admissible from a choice set, even when choice sets include all mixtures are:

- **coherent for *closed* (convex) sets of probabilities (Walley, 1990)**
- **and are generally *not* coherent otherwise (SSK&L, 2003).**

That is, such choice functions, ones that reduce to binary comparisons, conflate all the different sets of probabilities that meet the same set of supporting hyperplanes. These sets of probabilities are different at their boundaries only.

- **As we see next, these different (convex) sets of probabilities – though they share all the same supporting hyperplanes – are distinguished one from another by what they make admissible in non-binary choices.**

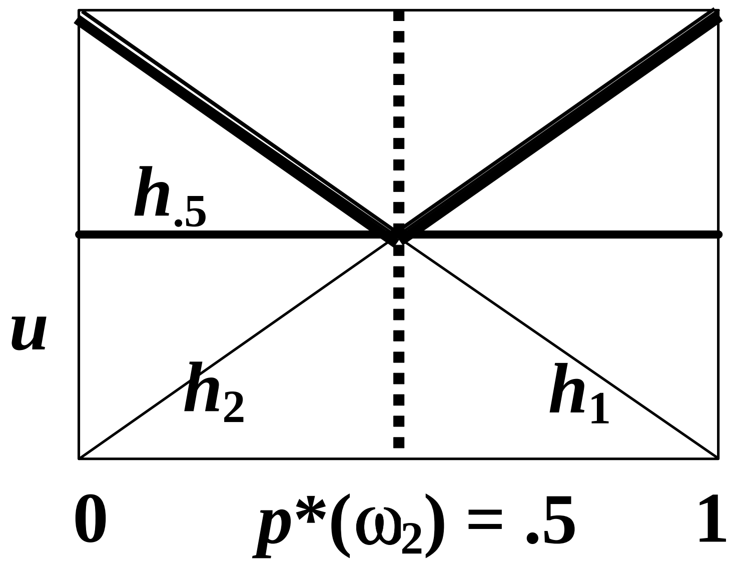
2.3 Using a coherent choice function to distinguish between sets of probabilities

The following toy-example illustrates how to use a coherent choice function to "test" for the presence/absence of a specific distribution $p^* \in P$ when S = $P \otimes u$, regardless the nature of the set P.

Return to the example of the coin with an unknown bias for landing *heads* (ω_2). Consider now the choice problem with these three options

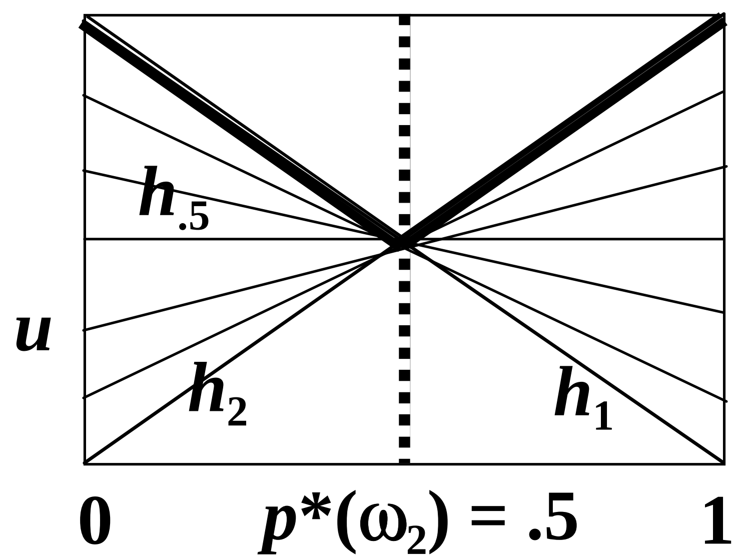
> **Let $h_{.5}$ be the constant horse lottery that pays off α = .5 in either state.**
>
> **Let h_1 be the horse lottery that pays w if the *heads* and b if *tails*.**
>
> **Let h_2 be the horse lottery that pays w if the *tails* and b if *heads*.**

We use the choice problem $O = \{h_{.5}, h_1, h_2\}$ to "test" for $p^* = (1/2, 1/2)$



- **$C[O] = \{h_{.5}, h_1, h_2\}$, all three options are admissible, if and only if $p^* \in P$.**

- **And if we close the option set under convex combinations, the same holds.**



The entire (convex) set of mixed options generated by h_1 and h_2 is admissible ***if and only if*** **$p^* \in P$.**

- **This technique generalizes to a "test" for an arbitrary distribution *p* on an arbitrary (finite) Ω against an arbitrary set *P*, as shown in the *Appendix*.**

3. *A representation of coherent choice functions under uncertainty*.

The Anscombe-Aumann SEU theory of preference on horse lotteries is given by these 4 axioms:

<u>AA-1</u> Strict preference $<$ is a *weak order* over pairs of horse lotteries.

$<$ *is anti-symmetric, transitive, and non-preference* (~) is transitive indifference.

<u>AA-2</u> *Independence* For all h_1, h_2, h_3, and $0 < \alpha \leq 1$.

$$h_1 < h_2 \text{ if and only if } \alpha h_1 \oplus (1-\alpha)h_3 < \alpha h_2 \oplus (1-\alpha)h_3.$$

AA-3 **An *Archimedean* condition**

For all $h_1 < h_2 < h_3$ there exist $0 < \alpha, \beta < 1$,

with $\alpha h_1 \oplus (1-\alpha)h_3 < h_2 < \beta h_1 \oplus (1-\alpha)h_3$.

AA-4 *State-independent utility*. In words, whenever ω is non-null, between two horse lotteries that differ solely in that one yields *b* in ω where the other yields *w*, the agent prefers the former act.

In the language of this presentation, the Anscombe-Aumann *SEU* Theorem is that such a binary preference over pairs of horse lotteries is represented by a coherent choice function, with *P* a singleton set comprised by one probability distribution *p* over Ω.

The first AA-1 axiom is that preference is a weak order. We can use Sen's result to reformulate AA-1, so that the corresponding choice function satisfies *Properties alpha* and *beta*:

- **Property *alpha*: You cannot promote an inadmissible option into an admissible option by adding to the choice set of options.**
- **Property *beta*: If two options are both admissible from some choice set, then whenever both are available, either both are admissible or neither one is.**

Then define the (strict) preference relation between two options $h_1 < h_2$ to mean that $C[\{h_1, h_2\}] = \{h_2\}$, i.e., only h_2 is admissible from the pair, $\{h_1, h_2\}$.

The rest of the axioms are easily expressed in terms of choice functions.

In our setting, we can generalize Anscombe-Aumann theory to accommodate coherent choice functions, as follows. I will express the axiom in words. Each is evidently necessary for a choice function to be coherent.

Structural Assumption on sets of options:

Each option set is *O* is *closed*, to insure that admissible options exist.

- **Axiom 1a: *Property alpha* – you can't promote an inadmissible option into an admissible option by adding to the option set.**
- **Axiom 1b: You cannot promote an inadmissible option into an admissible option by deleting inadmissible options from the option set.**

With A1a and A1b, define a strict partial order $\langle$ on sets of options as follows. Let O_1 and O_2 be two option sets. Recall that $R[\bullet]$ is the rejection function associated with the choice function $C[\bullet]$.

Defn: $\quad O_1 \langle O_2$ ***if and only if*** $\quad O_1 \subseteq R[O_1 \cup O_2]$.

That is, it follows from Axioms 1a & 1b that $\langle$ is a strict partial order.

<u>Axiom 2</u> *Independence* is expressed for the relation $\langle$ over sets of options just as before.

$$O_1 \langle O_2 \quad \textit{if and only if} \quad \alpha O_1 \oplus (1-\alpha)h \ \langle\ \alpha O_2 \oplus (1-\alpha)h.$$

Let $H(O)$ be the result of taking the (closed) convex hull of the option set O. That is, H augments O with all its mixed options – and then close up the set.

<u>Axiom 3</u> *Convexity* If $h \in O$ and $h \in R[H(O)]$, then $h \in R[O]$.
Inadmissible options from a mixed set remain inadmissible even before mixing.

> **Note: Axiom 3 is needed to eliminate the two choice functions *Walley-Sen Maximality*, and *Γ-Maximin*, which are not coherent over general option sets, e.g., they are not always coherent when the option set fails to be convex or closed.**

Axiom 4: The Archimedean condition requires a technical adjustment, as the canonical form used by von Neumann-Morgenstern and Anscombe-Aumann theory is too restrictive in this setting (see SSK-95).

The reformulated version expresses the Archimedean condition as a continuity principle compatible with strict preference as a strict partial order. It reads, informally, as follows.

> **Let A_n and B_n (n = 1, …) be sets of options converging pointwise, respectively, to the option sets A and B. Let N be an option set.**

Axiom 4a: If, for each n, $B_n \langle A_n$ and $A \langle N$, then $B \langle N$.

Axiom 4b: If, for each n, $B_n \langle A_n$ and $N \langle B$, then $N \langle A$.

Axiom 5: The existence of a state independent utility for $\langle$ is unchanged from the Anscombe-Aumann theory.

Existence of an agreeing personal probability that extends the choice function C.

For distribution p on Ω, define the option sets $T_p = \{H_{1,p}, \ldots, H_{n,p}\}$ and $A_p = \{a_p, H_{1,p}, \ldots, H_{n,p}\}$ as in the *Appendix*, which generalizes section 2.3.

- ***Lemma*: There exists a non-empty, convex set of *target distributions* P_T such that for each $p \in P_T$, $R(T_p) = \varnothing$.**
- ***Lemma*: When the target set P_T has a face that is not open, there exists a probability $p \in P_T$ on this face such that $R(A_p) = \varnothing$ and then p consistently extends the choice function C[•].**

However, when the target set T_p has an open face, an additional axiom is needed to assure coherence of the choice function.

Axiom 6 **Where the convex target set T_p has a face that is open, there exists a sequence of distributions $p_n \in P_T$ converging to a distribution p on this face such that $R(A_{p_n}) = \varnothing$, (n = 1, 2, …).**

***Main Result*: A choice function under uncertainty is coherent *if and only if* it satisfies these 6 axioms.**

Corollary **(see the Appendix): The coherent choice function associated with the set *P* of probabilities on Ω is unique to *P*. Different sets *P* yield different choice coherent choice functions.**

The emphasis here is on the fact that the set *P* of probabilities used to represent the coherent choice function is entirely arbitrary. There is no assumption that *P* is closed, or convex, or even that it is connected.

Summary and principal conclusions regarding coherent choice functions when the decision maker's degrees of belief are given by a set of probabilities.

- **Some choice functions, e.g., *Maximality*, are coherent only in special circumstances, as when the option set *O* is convex and a *closed* set of probabilities *P* is used to represent uncertainty.**

- **Coherent choice functions based on binary comparisons between options fail to distinguish among different convex sets of probabilities, all of which share the same supporting hyperplanes.**

- **This restriction does not apply to all coherent decision functions: e.g., Levi's *E-admissibility* avoids this limitation.**

- **A representation exists for coherent choice functions under uncertainty, modeled on the A-A theorem. It applies to each set *P* of probabilities.**

- **Coherent choice functions are capable of distinguishing between each two different sets of probabilities, regardless of their formal structure.**

Appendix – A decision problem that identifies a particular target distribution.

Consider a finite state space $\Omega = \{\omega_1, \ldots, \omega_n\}$ with horse lotteries defined on prizes 1 and 0, with U(1) = 1 and U(0) = 0, in each state.

Let *P* be a set of probabilities. Let *R* be the Bayes rejection function related to *P*.
Let p = $(p_1, \ldots, p_n)$ be a probability distribution on Ω .
Let $\underline{p}$ be the smallest nonzero coordinate of p.
***Define* the constant horse lottery act $a = \underline{p}1 + (1 - \underline{p})0$.**
For each $j = 1, .., n$, *define* the act H_j by

$$
\begin{aligned}
H_j(\omega_i) &= 1 && \text{if } i = j \text{ and } p_j = 0,\\
&= a && \text{if } i \neq j \text{ and } p_j = 0,\\
&= (\underline{p}/p_j)1 + (1 - \underline{p}/p_j)0 && \text{if } i = j \text{ and } p_j > 0,\\
&= 0 && \text{if } i \neq j \text{ and } p_j > 0.
\end{aligned}
$$

Define the option set $A = \{a, H_1, \ldots, H_n\}$.

- ***Theorem:*** **p $\in$ *P* *if and only if* R(*A*) = $\varnothing$.**

***Corollary* – a generalization of the theorem that appears at the end of our paper, "A Rubinesque theory of decision" (2004).**

Let P_1 and P_2 be two distinct (nonempty) sets of probabilities with corresponding Bayes rejection functions R_1 and R_2.

There exists a finite option set *A* as above such that $R_1(A) \neq R_2(A)$.

Thus, each set of probabilities *P* has its own distinct pattern of Bayes rejection functions with respect to option sets A_p for p $\in$ *P*.

Selected References

Anscombe, F.J. and Aumann, R.J. (1963) "A definition of subjective probability" *Ann Math. Statist*. 34: 199-205.

Gilboa, I. and Schmeidler, D. (1989) "Maxmin expected utility with non-unique prior," *J. Math Econ* 18: 141-153.

Kadane, J.B., Schervish, M.J., and Seidenfeld, T. (2004) "A Rubinesque theory of decision," *IMS Lecture Notes Monograph* 45: 1-11.

Levi, I. (1974) "On indeterminate probabilities" *J.Phil*. 71: 391-418.

Levi, I. (1980) *The Enterprise of Knowledge*. MIT Press.

Schervish, M.J., Seidenfeld, T, Kadane, J.B., and Levi, I. (2003) "Extensions of expected utility theory and some limitations of pairwise comparisons" In *Proceedings of the Third International Symposium on Imprecise Probabilities and Their Applications* (J-M Bernard, T.Seidenfeld, and M.Zaffalon, eds.) 496-510. Carleton Scientific.

Seidenfeld, T., Schervish, M.J., and Kadane, J.B. (1995) "A representation of partially ordered preferences," *Ann Stat*. 23: 2168-2217.

Sen, A. (1977) "Social choice theory: a re-examination," *Econometrica* 45: 53-89.

Smith, C.A.B. (1961) "Consistency in statistical inference and decision," *J.Roy.Stat Soc*. Ser B 23: 1-25.

Walley, P. (1990) *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London.

*Research for this work was supported by NSF grant DMS-0139911, and was partially supported by the Alexander von Humboldt Foundation, the Federal Ministry of Education and Research, and the Program for the Investment in the Future (ZIP) of the German Government through a Sofja Kovalevskaja Award.

Addendum – On the (uniform) strict dominance of non-Bayes decisions: an argument in the fashion of D. Pearce (1984), Lemma 3, p. 1048.

Let $\Omega = \{\omega_1, \omega_2, \ldots\}$ be a partition of states, indexed by the set J. Let A be a class of acts defined on Ω, indexed by the set I, such that for $a_i \in A$, $a_i(\omega_j) = u_{ij}$, a cardinal utility of the consequence of a_i when state ω_j obtains. Assume that utilities are bounded above and below.

Let Q be the class of finitely additive probability distributions over Ω with a "prior" over states denoted q. Similarly, let P be the class of finitely additive mixtures over A, with mixed acts denoted p.

Since expectations of a finitely additive mixed act p with respect to a finitely additive prior q, for $(p, q) \in P \otimes Q$, depend generally on the order of integration, we define these expectations with respect to a parameter, w, $0 \leq w \leq 1$, as follows.

$$E^w[p,q] = w\int_j \int_i u_{ij}\, dp(a_i)dq(\omega_j) + (1-w)\int_i \int_j u_{ij}\, dq(\omega_j)dp(a_i).$$

Note that in case p or q is countably additive, $E^w[p,q]$ does not depend upon w. Consider the case with $w = 0$.

$$E[p,q] = \int_i \int_j u_{\mathbf{ij}}\, \mathrm{d}q(\omega_j)\mathrm{d}p(a_i). \qquad \textbf{(1)}$$

This choice of w produces a value for decisions that is least favorable to the decision maker. See Theorem 2.2 S-S (1996).

Theorem: **Suppose that for each $q \in Q$, act $a \in A$ fails to maximize expected utility. That is, $a \neq$ arg max$_A$ $E[a,q]$.**

1. **There is a mixed alternative p^* that uniformly, strictly dominates a**

   $U(p^*(\omega_j)) > U(a(\omega_j)) + \varepsilon$, **for $j = 1, \ldots.$, with $\varepsilon > 0$,**

   and where p^* maximizes expected utility for some $q^* \in Q$.

2. **There is a *finite* mixture $p^o = \Sigma\alpha_i a_i$ ($\alpha_i > 0$, $\Sigma\alpha_i = 1$, $a_i \in A$, and $i = 1, \ldots, n$) where p^o uniformly, strictly dominates a. That is**

   $U(p^o(\omega_j)) > U(a(\omega_j)) + \varepsilon$, **for $j = 1, \ldots.$, with $\varepsilon > 0$.**

- **Note that with this result we are able to apply the strict standard of deFinetti's "incoherence" (= uniform, strict dominance) to a broad class of decisions under uncertainty, analogous to traditional Complete Class Theorems for Bayes decisions. The standard of incoherence used here is notably stronger than the mere *inadmissibility* (= weak dominance) of non-Bayes decisions, as is used in those Complete Class theorems.**

Proof. **(1) By hypothesis, assume that for each $q \in Q$, a fails to maximize expected utility with respect to the options in A. So, let $f(q) = a^q$ such that $E[a^q,q] > E[a,q]$.**

Transform the decision problem to one where the new utility U' is "regret" with respect to a. That is, for each $a_i \in A$, $U'(a_i(\omega_j)) = u'_{ij} = u_{ij} - U(a(\omega_j))$. Hence, $U'(a(\omega_j)) = 0$, for $j = 1, \ldots$. Evidently, U' is bounded, as U is.

Consider the two-person, zero-sum game in which player-1's payoff for choosing act a_i when player-2 chooses state ω_j is u'_{ij}, and player-2's payoff is $-u'_{ij}$. As before, define $E'[p,q] = \int_i \int_j u'_{ij}\, dq(\omega_j) dp(a_i)$ as the value to player-1 when the pair

of mixed strategies (*p*,*q*) is played. Thus, for each $q \in Q$ and for each pair of mixed acts $p, p' \in P$, $E'[p, q] \geq E'[p', q]$ if and only if $E[p, q] \geq E[p', q]$, since $E'[p, q] = E[p, q] - E[a, q]$.

According to Theorem 3 of Heath and Sudderth (1972), the von Neumann Minimax Theorem obtains within the space of finitely additive mixed strategies. That is, the game has a value V where:

$$\inf_Q \sup_P E'[p,q] = \sup_P \inf_Q E'[p,q] = V.$$

Moreover, as an easy consequence of this result – see Theorem 2.1 of Kindler (1983) – there are good strategies for each player, $(p^*, q^*) \in P \otimes Q$, that achieve the game's value V. That is

$$\inf_Q E'[p^*, q] = \sup_P E'[p, q^*] = V.$$

Now observe that $E'[p^*, q^*] \geq E'[a^{q^*}, q^*] > E'[a, q^*] = 0.$

The first inequality is because p^* **is a best response for player-1 to player-2's choice of** q^***. The second inequality is because** a^{q^*} **is a strictly better response to** q^* **than is** a**.**

Note that since q^* **is player 2's best response to player-1's choice of** p^***, we have that for each** $q \in Q$**,** $E'[p^*, q] \geq E'[p^*, q^*]$**. Thus,**

$$E'[p^*, \omega_j] \;\geq\; E'[p^*, q^*] \;\geq\; E'[a^{q^*}, q^*] \;>\; E'[a, \omega_j] \;= 0.$$

This establishes that p^* **uniformly strictly dominates** a **across states, as the difference, expressed either in** U **or** U'**, is never less than** $E'[a^{q^*}, q^*] > 0$**, and** p^* **maximizes expected utility against the f.a. "prior"** q^***.**

(2) Let P^o **be the set of mixed strategies for the decision maker that have finite support. Heath-Sudderth (1972, Theorem 3) also show that:**

$$\inf_Q \sup_{P^o} E'(p,q) \;=\; \sup_{P^o} \inf_Q E'(p,q) = V$$

Thus, for the player whose strategy is on the outside **of the order of integration in (1) the value of the game is the same whether taken over all of** P **or merely over the restricted subset** P^o **of** P**.**

Then, for each $0 < \delta$**, there is an element** p^o_δ **in** P^o **such that**

$$\inf_Q E'(p^o_\delta, q) \geq E'[p^*, q^*] - \delta \geq E'[a^{q^*}, q^*] - \delta.$$

Hence, $E'[p^o_\delta, \omega_j] \geq E'[p^*, q^*] - \delta \geq E'[a^{q^*}, q^*] - \delta > E'[a, \omega_j] - \delta$**.**
Choose $\delta < E'[a^{q^*}, q^*]$ **and then** $E'[a^{q^*}, q^*] - \delta > 0 = E'[a, \omega_j]$.

So, $E'[p^o_\delta, \omega_j] > E'[a, \omega_j]$**, and in the original decision problem** p^o_δ **also dominates the non-Bayes option** a**, state-by-state, by at least** $E'[a^{q^*}, q^*] - \delta$**.**

Additional References for the Addendum

Heath, D. and Sudderth, W. (1972) "On a Theorem of deFinetti, Oddsmaking, and Game Theory," ***Annals Math Stat*****. 43: 2072-2077.**

Kindler, J. (1983) "A General Solution Concepts for Two-Person, Zero-Sum Games," ***J. Optimization Theory and Applications*** **40: 105-119.**

Pearce, D. (1984) "Rationalizable Strategic Behavior and the Problem of Perfection," ***Econometrica*** **52: 1029-1050.**

Schervish, M.J. and Seidenfeld, T. (1996) "A Fair Minimax Theorem for Two-Person (Zero-Sum) Games Involving Finitely Additive Strategies" in Bayesian Analysis in Statistics and Econometrics. D.Berry, C.Chaloner, and J.Gwerke (eds.) New York: John Wiley, pp. 557-568.