Comment on Teddy Seidenfeld's "Coherent Choice Functions Under Uncertainty"
Formal Epistemology Workshop, 2005
Katie Steele, University of Queensland

I will use this commentary to further illustrate how the choice function Seidenfeld et al. (2004) characterize with their six axioms compares with potential alternative choice functions under uncertainty, and under what conditions are there distinctions.

Consider the following decision scenario (illustrated in Figure 1 below):

Let w_1 be the case that a coin lands tails and w_2 the case that it lands heads.
The bias of the coin is unknown, such that the set of possible probability distributions under consideration, $S = \{0 \leq p(w_1) \leq 1\}$.

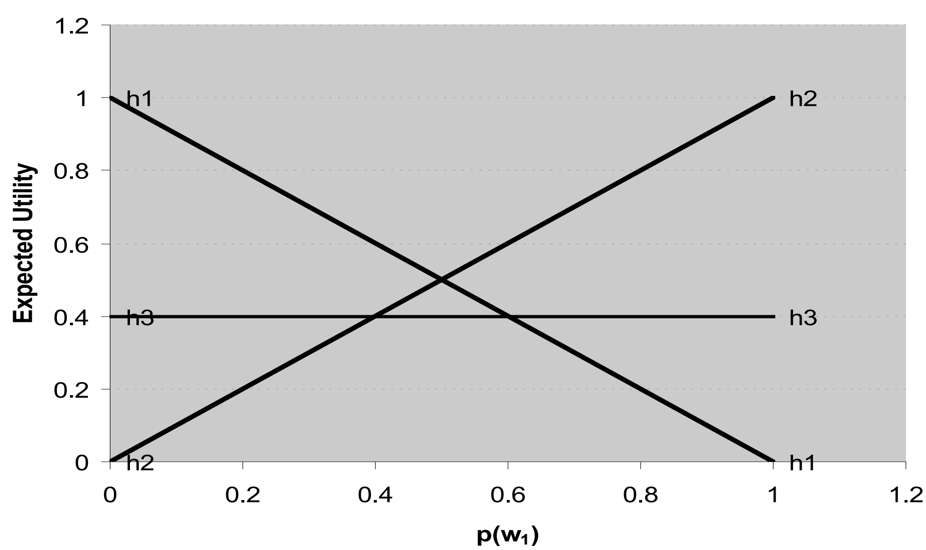
The option set $O = \{h_1, h_2, h_3\}$

Where

h_1 is the horse lottery paying 1 if the coin lands tails, and 0 if it lands heads
h_2 is the horse lottery paying 0 if the coin lands tails, and 1 if it lands heads
h_3 is the constant horse lottery paying 0.4 whether the coin lands heads or tails.

The (imprecise) expected utilities of the options (i.e. the expected utility of the options for each possible probability distribution) are depicted in Figure 1.

Figure 1 : EU for lottery options over the set of probability distributions



This example is intended to show the significance of the constraint on the choice function that the option set should be considered closed under mixtures, or rather that admissible options should remain admissible if the option set *were to be* closed under mixtures. This constraint is formalized by the 3rd axiom (Seidenfeld FEW 2005 slides; p. 23):

Convexity: If $h \in O$ and $h \in R[H(O)]$, then $h \in R[O]$

Where

R is the set of rejected options (as per Seidenfeld's terminology)
H(O) is the result of taking the (closed) convex hull of the option set O.

(Inadmissible options from a mixed set remain inadmissible, even before mixing.)

Briefly the choice function Seidenfeld et al. (2004) characterize identifies the Bayes-solutions: it makes an option admissible if it maximizes expected utility for at least one probability-utility pair <p, u> in the set S of all such permissible pairs.

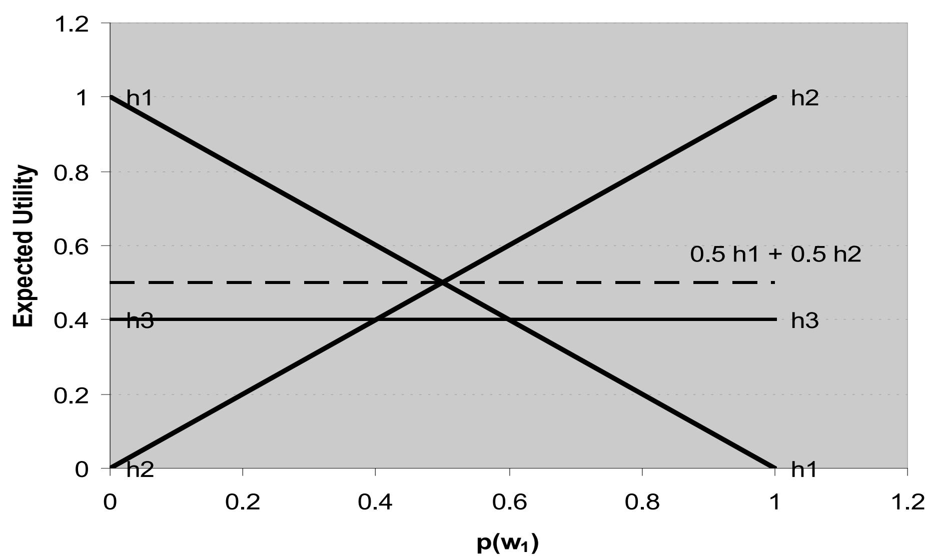
Referring to the example in Figure 1, only h_1 and h_2 are admissible by this choice function. h_3 is not admissible because it does not maximize expected utility for any <p, u> pair in S. Clearly the choice function operates over sets and is not reducible to a pair-wise comparison of options: If h_3 is compared with either of the alternatives h_1 or h_2 *alone* it is admissible, because with respect to either h_1 or h_2 it has maximal expected utility for at least some <p, u> pair in S. That is, neither h_1 nor h_2 strictly dominates h_3, i.e. neither h_1 nor h_2 has greater expected utility than h_3 for all permissible probability distributions. When all three options are considered together, however, h_3 is not admissible, because it does not have maximal expected utility for any <p, u> pair in S.

If we confine our attention to the option set described in our example, $O = \{h_1, h_2, h_3\}$, when it is *not* closed under mixtures, Seidenfeld et al.'s choice function identifying the Bayes-solutions is not entirely uncontroversial. Some may prefer a more "minimal" pair-wise rule that discards an option only if it is strictly dominated by another alternative. This is the spirit of the "strengthened" Kyburg rule (see Seidenfeld 1983), or Walley-Sen's principle of Maximality (Walley 1990). Such a choice function would not rule out any of the options in our option set (in the absence of mixed options). Furthermore, it may well be argued that rationality should permit the precautionary maxi-min rule, which amounts to choosing the option that has the greatest minimum expected utility over all <p, u> pairs. For our example (again, in the absence of mixed options), the maxi-min choice would be h_3, because it has a minimum of 0.4, whereas both h_1 and h_2 have minimum expected utilities of 0. As noted, h_3 is precisely the option that is excluded by Seidenfeld et al.'s choice function.

The point I am making is that, in the absence of mixed options, Seidenfeld et al.'s choice function is not uncontroversial because alternative admissible option sets cannot be ruled out by a Dutch Book argument. This is not to say that alternative choice functions, such as Walley-Sen Maximality or the maxi-min rule, are not without their problems. In particular, let us now consider the situation where mixed options are taken into account. Pair-wise choice functions can give different results when this condition is imposed. It turns out that when option sets explicitly include all mixed options, Walley-Sen Maximality (our standard for pair-wise reasoning) coincides with Seidenfeld et al.'s choice function, where it did not necessarily coincide without the inclusion of mixed options. In fact, Seidenfeld proves that if a solution is not Bayes, there will be a mixed option that strictly dominates it (Seidenfeld FEW 2005 slides; pp. 32–7).

For our example, graphing just one mixed option (as in Figure 2) shows that h_3 is pair-wise strictly dominated when the option set is closed under mixtures:

Figure 2 : EU for lottery options (including a mixed option) over the set of probability distributions



So the question is whether we want a choice function to be consistent when it is closed under mixtures (as per Axiom 3). Seidenfeld et al.'s choice function identifying the Bayes-solutions is consistent in this way; pair-wise rules are not. It is surely a virtue of a choice function to select only options that would be admissible even if mixed options were included. After all, mixed options are commonly appealed to in game theory, and in theory we always have the capability of following a mixed option, rather than a pure option. Nevertheless, the point might be pressed that in real situations, there is often not the capability for acting on a mixed option, and that a rational choice function need only operate on the options that are actually specified in the option set.

Even when if we restrict our attention only to option sets that are closed under mixtures, there is divergence between Seidenfeld et al.'s choice function and the pair-wise Walley-Sen principle of Maximality when probability sets are not convex (Seidenfeld FEW 2005 slides, p. 14). Seidenfeld et al.'s axioms allow non-convex probability sets for reasons that will be discussed later. The possibility for such divergence calls for further defense of a choice function that operates over sets, as opposed to a pair-wise choice function. Seidenfeld (FEW 2005 slides, pp. 15–7, 29–30) offers such a defense; his illustration is sophisticated and general, involving a proof that applies to decision situations in which the sets of probability functions differ. I will illustrate the gist of this argument, using a much simpler and isolated example.

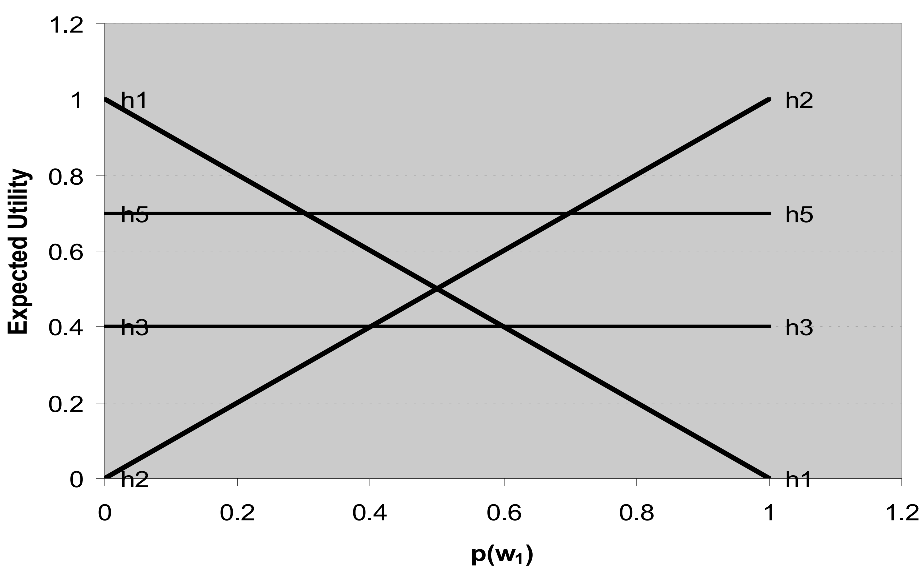
In brief, Seidenfeld proves that his choice function operating over sets is capable of making more distinctions between decision scenarios when compared to a pair-wise rule. I want to illustrate what is meant here by a choice function that makes "more distinctions between decision scenarios". I will use the following (admittedly limited) example:

Consider two decision scenarios with the following option sets:

$O_1 = \{h_1, h_2, h_3\}$
$O_2 = \{h_1, h_2, h_5\}$

Where h_1, h_2 and h_3 are defined as before and h_5 is the lottery giving a constant payoff of 0.7. Ignore mixed options for this example. All the options are depicted in the following graph (Figure 3):

Figure 3 : EU for lottery options over the set of probability distributions



For both of the decision situations in the example above, all options are admissible by the pair-wise Walley-Sen principle of Maximality. On the other hand, the Bayes-solutions for scenario 1 are $\{h_1, h_2\}$ and for scenario 2 $\{h_1, h_2, h_5\}$. In other words, Seidenfeld et al.'s choice function here makes a distinction between the option set that contains h_3 and the option set that contains h_5. As mentioned, Seidenfeld proves that a greater ability (when compared to a pair-wise rule) to make distinctions between decision situations is a general characteristic of the choice function. But whether or not this lends support to the choice function depends on whether one thinks making more rather than less distinctions between different decision situations is a virtue of a rational choice function. It is surely a useful characteristic of a choice function, but I don't think it is self-evident that rationality adjudicates in favour of more rather than less distinctions.

Finally, given that there is disagreement between Seidenfeld et al.'s choice function and alternative pair-wise rules for non-convex probability sets, I would like to say something about why it is important to consider the non-convex case. The importance of this generalization is I think best highlighted by the social choice application. Indeed, social choice situations are a good example of when we might reasonably have sets of probability and utility distributions, rather than a singleton pair, because each individual in the group may nominate a distinct probability and utility distribution for the option outcomes. Of course there are numerous ways to resolve the social choice problem, but one plausible method is to consider these individual probability and utility distributions as members of a set of <p, u> pairs that together characterize the decision problem.

Now Levi's E-admissibility rule (Levi, 1974), like Seidenfeld et al.'s choice function, selects the Bayes-solutions to a decision problem, but it stipulates that the set of <p, u> pairs be convex. For the social choice application, this effectively means that all mixes and matches of the <p, u> pairs contributed by group members are taken into account, including all probability and utility values lying in between individuals' estimates.

Besides the fact that it is questionable (from an intuitive standpoint) as to whether the convex set representation of members' probability and utility evaluations is a good model of the social choice scenario (e.g. why do we need to consider <p, u> pairs that are not identified with any particular individual?), it is shown mathematically to have undesirable features. Importantly, Levi's model does not respect the Pareto preferences of the group: it can be shown that the only E-admissible orderings that preserve Pareto preferences are the trivial ones where a single member acts as dictator (Seidenfeld et al. 1989).

Seidenfeld et al.'s axioms characterize the more general choice function in that it allows for non-convex sets of <p, u> pairs. This has the nice feature that for social choice situations, if the <p, u> pairs contributed by members are not mixed and made convex, then the Pareto preferences of the group are respected. I think this is good reason to consider non-convexivity of the set of <p, u> pairs as an important generalization of Levi's E-admissibility rule.

Works Consulted

Levi, Isaac. 1974. On Indeterminate Probabilities. *Journal of Philosophy* 71 (13):391-418.

Seidenfeld, Teddy. 1983. Decisions with Indeterminate Probabilities. *The Behavioral and Brain Sciences* 6:259-261.

Seidenfeld, Teddy, Joseph B. Kadane, and Mark J. Schervish. 1989. On the Shared Preferences of Two Bayesian Decision Makers. *Journal of Philosophy* 86 (5):225-244.

Seidenfeld, Teddy, Joseph B. Kadane, and Mark J. Schervish. 2004. A Rubinesque theory of decision. *IMS Lecture Notes Monograph* 45:1-11.

Walley, Peter. 1990. *Statistical Reasoning with Imprecise Probabilities*. London: Chapman and Hall.