Comments on John Mumma, "Proofs, Pictures, and Euclid"

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John Mumma is engaged in an interesting project to reconstruct Euclid's proofs in a manner more charitable than has been traditional for the past few centuries. Since Hilbert's fully modern axiomatization of geometry (which he proved to be equivalent to Cartesian geometry with coordinates chosen from a field with certain closure conditions), it has been assumed that Euclid's proofs were very faulty and incomplete. Mumma's suggestion (following Manders and others) is that by allowing the diagrams to play an equal role in the proofs, rather than considering them only illustrative of the text, much of this incompleteness can be rectified. There are still problems - in the first proposition, constructing an equilateral triangle on a given segment, Euclid uses, without explicit proof or assumption, the fact that two circles passing through each other's center must intersect, and that two lines intersect in at most one point. However, in Mumma's system it is only these further assumptions that must be made explicit, rather than reformulating everything in a very different language the way Hilbert would require.

An important worry historically that motivates the thought that diagrams cannot play an essential role in proof is the fact that diagrams always have specific properties, even if the theorem to be proved is general. For instance, in a proof that the interior angles in any triangle always add up to two right angles, the diagram will either portray the triangle as acute, right, or obtuse, even though the proof is supposed to apply to all three. One might suppose that as long as these special properties of the diagram are never used, the proof will hold generally, but it seems unclear what role the diagram can play if these special properties aren't used, and how to draw the line between which properties are special and which are general.

Ken Manders has suggested that this dividing line is between the properties he calls "exact" (which can't be used) and "co-exact" (which can). However, we must still be careful - a famously fallacious proof that all triangles are isosceles relies on the diagram only for these co-exact properties. (See Mumma's paper for the proof.) Mumma's contribution as I understand it is to point out that constructions of the diagram are steps in the proof just as much as the textual inference is. Instead of relying on all co-exact properties, we can only rely on properties that are generated by relations among "prior" elements in the construction. The diagram for the fallacious proof relies on a specific construction having already been carried out, with the dependencies ignored. If we try to carry it out, we see that the angle bisector and the perpendicular bisector have no relation of direct dependency, so we must be careful in combining the diagrams to consider different cases when drawing inferences. A fuller presentation of Mumma's system will have to specify just how things work in a situation like this.

As Mumma points out, there are cases in which inferences are legal according to his system, but apparently not supported by the diagram. For instance, if the underlying array of points is too coarse, then various intersection points may not appear. This is solved by considering an equivalence class of equivalent diagrams, rather than just a single diagram. However, in a diagram with six points A, A', B, B', C, C', we will have to be more careful - different members of the equivalence class will give different orderings of points when we extend them by constructing segments AA', BB', and CC', and compare their intersection points. This is just as it should be, but it may make us worry a bit about just what conditions allow any sort of construction at all.

The equivalence classes present another worry - as equivalence classes, they bear relatively little resemblance to the diagrams Euclid provides in the Elements. It is bad enough that points always lie on a grid, and that circles are polygons, but it's even worse now that everything is really an equivalence class of these sorts of elements. These diagrams now appear to be abstract mathematical objects rather than concrete pictures. Although these abstract diagrams may allow for shorter proofs than Hilbert's formal system, and may also be sound and complete, it's not totally clear in what sense they explicate Euclid's proofs if they're not the sort of thing that can actually be drawn.

Admittedly, a similar worry arises for sentential reconstructions of mathematical proofs too. Mathematicians produce concrete tokens of ink on paper or chalk on a board, and leave out many steps. If traditional Hilbert-style proofs can be considered adequate reconstructions of some of these informal proofs, then perhaps these abstract diagrams can be considered adequate reconstructions of Euclid's proofs.

A more pressing concern than whether abstract objects can play this role is whether geometric objects can. If the proofs are to play some sort of justificatory role for our geometric knowledge, then it seems plausible that one should be able to understand them without yet understanding geometry. However, if part of the proof is itself geometric, then it looks like there is some kind of circularity.

I think the natural response here is to point out that the diagrams Mumma uses are much simpler than the actual geometric configurations being reasoned about - this is where the use of polygons for the circles makes things work better. Since the diagrams are based on finite arrays of points, there are finitary arithmetical procedures that allow us to check when lines intersect and the like. We need something like this much arithmetic anyway to formalize the linguistic symbols used in Hilbert-style proofs, so this might be reasonable. In this sense, the diagrams aren't geometric objects at all, but much simpler syntactic objects, just like the sentences in standard formal proofs.

Of course, any suggestion that makes the diagrams non-geometric seems not to do justice to the notion that these are *diagrams* rather than any other sort of syntactic entity. A response that preserves the geometric nature of the diagrams might stress their concreteness, but this would seem to bring back the worries about whether such proofs can be sufficiently rigorous.

A final worry might question the purpose of the whole project. We already know that much mathematical reasoning from the past was woefully incomplete. Newton, Leibniz, Euler, and others were able to gather a great deal of mathematical knowledge despite not being able to provide proofs that are up to modern standards. As anyone who has taken a math class knows, modern students are also able to gain a lot of knowledge without possessing complete proofs. One might be able to argue in the contemporary case that this knowledge is parasitic on testimony as to the existence of such a proof, but this is unavailable in the historical case. There have been attempts to use the infinitesimals of Abraham Robinson to rehabilitate the methods of the early modern analysts, but these have not succeeded. Why should we have any more reason to believe that the methods of the ancient geometers can be rehabilitated?

However, Mumma's project does not appear to be quite as ambitious as this - he doesn't intend to rehabilitate Euclid's proofs completely. He realizes that Euclid has left out assumptions guaranteeing existence and uniqueness of certain types of intersections. However, he wants to show how much of Euclid really can be reconstructed on rigorous grounds, using the diagrams as integral parts of the proof. In this sense, it seems that there are two goals - one is historical, to show that Euclid was closer to modern standards than we might have thought; the other is epistemic, to show that diagrams (as opposed to linguistic symbols) can in fact play this sort of role.

The latter point would then generalize to diagrams in other areas of mathematics, though Mumma admits that there is still work to be done in distinguishing which diagrams play this important epistemic role, and which play a more heuristic role. Mumma gives examples of diagrams from texts in modern geometry, topology, and analysis that obviously play no role in the proof. However, there are diagrams in category theory that are clearly intended as part of their proofs. Diagrams in knot theory and some other areas seem to possibly lie somewhere in between. It would be very interesting to see in which of these areas, and to what extent, a project like Mumma's can succeed.