Quantified Logic of Awareness and Impossible Possible Worlds

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Abstract

Among the many possible approaches to the problem of logical omniscience, I consider here awareness and impossible worlds structures. The former approach, pioneered by Fagin and Halpern, distinguishes between implicit and explicit knowledge, and renders the agents immune from logical omniscience with respect to explicit knowledge. The latter, first developed by Rantala, allows for the existence of logically impossible worlds to which the agents are taken to have "epistemological" access; since such worlds do not behave consistently, the agents's knowledge is fallible relative to logical omniscience. The two approaches are known to be equally expressive in the propositional case with Kripke semantics. In this paper I show that the two approaches are equally expressive in the propositional case with neighborhood semantics. Furthermore, I provide predicate systems of both awareness and impossible worlds structures interpreted on neighborhood semantics and prove the two systems to be equally expressive.

Introduction

One of the contributions of this paper consists of the formal comparison between a first-order version of Fagin and Halpern's logic of awareness[1], on the one hand, and a version of Rantala's quantified epistemic logic interpreted over impossible worlds structures, on the other. The semantics of both systems are here given by *neighborhood models*, following up on the work of Arló-Costa (cf. [AC02]) and Arló-Costa and Pacuit (cf. [ACP06]). One of the motives of interest in modeling epistemic logic with neighborhood structures, as argued in [ACP06], lies in the fact that it allows us to use constant domains without committing to the validity of the Barcan formulas. This way, the use of neighborhood semantics makes it possible

[1] Note that in the original article [FH88] this logic is referred to as "logic of *general* awareness."

to interpret the modal operators as high-probability operators without incurring in Kyburg's "lottery paradox[2]." Interpreting epistemic logic over neighborhood structures is also appealing because, in so doing, one can significantly weaken the incidence of the logical omniscience problem[3]. In the case of the minimal classical system **E**, for example, the agents's knowledge is only closed under logical equivalence. However, systems only slightly stronger than **E**, albeit weaker than **K**, present serious instances of logical omniscience. Thus, although the weakest neighborhood models (in which only the modal axiom **E** is valid) reduce the impact of logical omniscience to a minimum, interesting applications of first-order neighborhood structures may suffer the problem of logical omniscience to an extent that justifies the introduction of further semantical devices meant to limit it. In this spirit, [Sil06] introduces and discusses a quantified system of awareness logic interpreted over neighborhood structures. Although it is a well-established result (cf, [Wan90], [FHMV95], [HP07]) that, at the propositional level, awareness structures (in short: AWA) and impossible possible worlds structures (in short: IPW) are equally expressive (in the sense that any knowledge ascription we can perform in AWA, we can also perform in IPW), such a result is lacking at the predicate level of analysis.

A further reason to explore the interrelation between first-order AWA and IPW lies at the intersection of both historical and philosophical motives. As to the former element, notice that the problem of logical omniscience was first identified (and, presumably, dubbed) in the classic monography by Hintikka [Hin62]. While the representation of propositional knowledge offered therein could be interpreted, semantically, as a harbinger of future propositional solutions to the problem of logical omniscience, the representation of knowledge involving quantifiers escaped for many years a satisfactory immunization against logical omniscience. Hintikka's efforts to effectively limit the agents' deductive abilities were intertwined with important philosophical questions about analyticity (cf. [Hin73a]) and culminated with his account of *surface semantics* in [Hin73b], which however was later found wanting by Hintikka himself in [Hin75]. In the same article, Hintikka claims that the culprit for the logical omniscience of the agents is to be identified with

[2] It was noted in [AC02] that if the modal operator is interpreted as high-probability, then Kyburg's paradox applies. For example, consider a lottery with 1,000 tickets. For each ticket x, x has high probability of being a loser. Applying the converse Barcan formula, it follows that with high probability *all* tickets are losers. In normal systems, the constant domain assumption validates the Barcan formulas, whereas in classical systems it need not do so.

[3] In normal epistemic systems, agents are *omniscient* (they know all logical truths) and *perfect reasoners* (they know all the consequences of what they know.) For a philosophical analysis of the problem, I refer the interested reader to the first section of [Sil07].

the logical consistency of the worlds they deem possible, and that Rantala's *urn models* ([Ran75]) provide a solution to the problem by introducing, loosely speaking, logically inconsistent worlds. In [Ran82b], Rantala turns urn models, and their game-theoretical semantics, into first-order Kripke models with impossible worlds. Since the IPW solution was first conceived and developed with respect to quantified epistemic systems, it seems thus in order that first-order AWA be compared and contrasted with IPW.

Besides the classic references mentioned above (for awareness logic, [FH88]; for impossible worlds, [Ran82a], [Ran82b]; for the equivalence between the two approaches, [Wan90], [FHMV95]), this work situates itself in the more recent literature on the topic as follows. The first-order logic of awareness used here was introduced in [Sil06], inspired by both Halpern and Rêgo's work on quantified propositional awareness [HR06] and by Arló-Costa and Pacuit's work on first-order classical models [ACP06]. Several approaches to the problem of logical omniscience are compared in [HP07], where the general conclusion is drawn that, the equivalence in expressivity notwithstanding, while awareness structure best capture the *modeler*'s point of view, impossible worlds semantics seem a better representation of the *agent*'s subjective status[4]. A considerable deal of research has been conducted by economists on representating the agent's awareness. Since standard Aumann structures (cf. [DLR99]) cannot represent unawareness, economists have turned their attention to non-standard structures, which are typically based on a lattice of state spaces ordered by their expressiveness and meant to capture different levels of agents' awareness (cf. [MR99], [HMS06], [Li06]). The approach based on impossible worlds has received less attention in the economics literature, with the exception of Lipman's application of the impossible worlds framework to decision theory in [Lip99].

The rest of this paper is organized as follows: in the first section, after briefly reviewing the standard propositional approach to modeling knowledge based on neighborhood structures, I introduce awareness and impossible worlds structures and show that the equi-expressivity results given in [Wan90], [FHMV95] and [HP07] for Kripkean structures carry over to neighborhood models. In the second section, I introduce the quantified awareness logics developed in [Sil06] and the quantified impossible world systems of [Ran82b] and show how they are related in terms of expressivity.

[4] However, in propositional epistemic systems with probability (as the ones considered in [Coz05] and [HP07]) there appears to be a difference in the expressiveness of awareness and impossible worlds models.

1 Propositional Systems

1.1 Classical systems of epistemic logic

The *syntax* of propositional epistemic logic[5] consists of a language $\mathcal{L}$ containing a countable set Φ of primitive propositions $p, q, r, \ldots$ closed under the two connectives $\wedge, \neg$ and the n modal operators $K_1, \ldots, K_n$. We denote the set of all formulas in $\mathcal{L}$ with $For_{\mathcal{L}}$. The *semantics* is based on neighborhood *frames* $\mathcal{F} = (W, \mathcal{N}_1, \ldots, \mathcal{N}_n)$, where[6] W is a set of possible worlds and each $\mathcal{N}_i$, with $i = 1, \ldots, n$ is a neighborhood function from W to the set of all sets of subsets of W. The idea is that, at each world w and for each agent i, the neighborhood function specifies the propositions that i knows at w, where with "proposition" is intended, intensionally, the set of all worlds in which the proposition holds. A neighborhood *model* $M = (\mathcal{F}, \pi)$ consists of a frame $\mathcal{F}$ and a valuation function $\pi : W \times \Phi \to \{0, 1\}$ which assigns a truth value to each atom in each world. Thus, the semantic clauses recursively defining the satisfiability relations are

$$(M, w) \models p \text{ iff } \pi(w, p) = 1$$

$$(M, w) \models \neg\varphi \text{ iff } (M, w) \not\models \varphi$$

$$(M, w) \models \varphi \wedge \psi \text{ iff } (M, w) \models \varphi \text{ and } (M, w) \models \psi$$

$$(M, w) \models K_i\varphi \text{ iff } \{v : (M, v) \models \varphi\} \in \mathcal{N}_i(w).$$

As a notational convention, we denote the set $\{v : (M, v) \models \varphi\}$ with $\|\varphi\|^M$, occasionally dropping the superscript when ambiguity does not arise, and refer to it, interchangeably, as the *truth set* or the *intension* of φ (relative, of course, to a model M.) We thus say that φ is *true* at w iff $(M, w) \models \varphi$; that φ is *valid* with respect to M iff φ is true in all worlds of the model irrespective of the valuation π, and denote such a fact with $M \models \varphi$; that φ is *valid* with respect to a class of frames $\mathcal{C}$ iff φ is true in every model based on a frame belonging to $\mathcal{C}$, and denote such a fact as $\mathcal{C} \models \varphi$.

We can characterize frames with respect to specific properties of neighborhoods. Different properties correspond to the validity of different axioms. In particular, since the truth sets of logically equivalent formulas are always identical, we have that the class of all neighborhoods frames validates axiom:

RE From $\varphi \leftrightarrow \psi$, infer $K_i\varphi \leftrightarrow K_i\psi$.

[5] For this exposition, cf. [Che80] and [ACP06].

[6] Note that, to avoid cluttering in the definitions, I shall occasionally use the abbreviation $\overrightarrow{x_i}$ (or even, if there is no danger of ambiguity, simply x_i) for the list of objects $x_1, \ldots, x_n$.

Frames characterized by the following properties (for each $X, Y \in \mathcal{P}(W)$ and $w \in W$):

(m) If $X \cap Y \in \mathcal{N}_i(w)$, then $X \in \mathcal{N}_i(w)$ and $Y \in \mathcal{N}_i(w)$;

(c) If $X \in \mathcal{N}_i(w)$ and $Y \in \mathcal{N}_i(w)$, then $X \cap Y \in \mathcal{N}_i(w)$;

(n) $W \in \mathcal{N}_i(w)$

validate, respectively, the following axioms:

M $K_i(\varphi \wedge \psi) \rightarrow K_i\varphi \wedge K_i\psi$

C $K_i\varphi \wedge K_i\psi \rightarrow K_i(\varphi \wedge \psi)$

N $K_i\top$,

and are said to be *supplemented*, *closed under intersections*, *possessed of the unit*, respectively.

The *minimal* classical system **E** contains the following axioms:

PC All tautologies of propositional calculus

RE From $\varphi \leftrightarrow \psi$, infer $K_i\varphi \leftrightarrow K_i\psi$

MP From $\varphi \rightarrow \psi$ and φ, infer ψ

The *monotonic* system **EM** adds axiom M to the axioms above; system **EMN** adds axiom M and N; etc. It is a well known fact[7] that system **EMCN** is pointwise equivalent to the weakest normal system **K**. In the following, I mostly confine my analysis to the monotonic system **EM**[8]

1.2 Awareness Structures

As stated in the introductory section, epistemic systems interpreted over neighborhood models still suffer the problem of logical omniscience, to some extent. Although such logics are weaker than the weakest normal logic **K**, they share with **K** many instances of logical omniscience. Axiom RE implies that the agents's knowledge is closed under logical equivalence. Axiom N implies that the agents know any tautology. Moreover, axioms RE and M together imply that agents's knowledge is closed under logical consequence: Let $\vdash \varphi \rightarrow \psi$; it follows (by PC) that $\vdash \varphi \leftrightarrow (\varphi \wedge \psi)$; by RE, that $\vdash K_i\varphi \leftrightarrow K_i(\varphi \wedge \psi)$; by M and PC, that $\vdash K_i\varphi \rightarrow K_i\psi$ so that, if $K_i\varphi$

[7] For a proof of it, the reader may consult the textbook [Che80].

[8] This restriction is justified by the large amount of applications, spanning across many areas, based on **EM** systems. **EM** systems augmented with awareness are proven to have a useful decidable fragment in [Sil07].

and ψ is a logical consequence of φ, it follows that $K_i\psi$. Thus, the use of a remedy to logical omniscience (as e.g. *awareness structures*) seems expedient not only when considering normal epistemic logics, but also when we are interpreting epistemic logic over neighborhood structures.

The idea behind awareness structures[9] is, simply put, that in order to be able to *actually* know a proposition, an agent first needs to be *aware* of that proposition. This formulation suggests that a distinction between two kinds of knowledge is at work. On the one hand, we have actual knowledge of a proposition φ by an agent who is in fact aware of φ, on the other we have knowledge of φ by an agent who is not aware that φ. In the influential [FH88], the former kind is called *explicit*, the latter *implicit* knowledge. While implicit knowledge is represented by the usual epistemic operators (therefore the agents are fully logically omniscient with respect to it,) explicit knowledge consists of the conjunction of implicit knowledge and awareness. Since the set of formulas of which agents are aware is, generally speaking, arbitrary, it follows that the agents are not logical omniscient with respect to explicit knowledge. For example, consider the argument from the previous paragraph showing that in the system **EM** the agents's knowledge is closed under logical consequence. While the argument remains valid with respect to the operators K_i representing implicit knowledge, we can however imagine a world w in which agent i is not aware of the proposition expressed by the formula φ. At w, agent i implcitly knows ψ, since ψ is a logical consequence of φ, but i does *not* explicitly know ψ, since she is not aware of it.

Formally, we supplement the syntax of $\mathcal{L}$ with n awareness operators $A_1, \dots, A_n$ and n explicit knowledge operators $X_1, \dots, X_n$, to obtain the language $\mathcal{L}^A$. Semantically, a propositional awareness structure is a tuple $M^A = (W, \mathcal{N}_1, \dots, \mathcal{N}_n, \mathcal{A}_1, \dots, \mathcal{A}_n, \pi)$, where $(W, \mathcal{N}_1, \dots, \mathcal{N}_n)$ is a standard neighborhood frame. To obtain a model, we assign a truth value to the atoms in each world by means of a standard valuation π, while we associate with each agent i, for each world w, an *awareness set* $\mathcal{A}_i(w)$ containing a subset of the formulas of $\mathcal{L}^A$. To take care of the new modal operators, we add to the definition of the satisfiability relation the following self-explanatory clauses:

$$(M, w) \models A_i\varphi \text{ iff } \varphi \in \mathcal{A}_i(w)$$

$$(M, w) \models X_i\varphi \text{ iff } (M, w) \models K_i\varphi \text{ and } (M, w) \models A_i\varphi.$$

Axiomatically, we can define the explicit knowledge operator in terms of

[9] The seminal reference on awareness is [FH88]. More recent work is Halpern's [Hal01], while the economics literature offers the accounts of Heifetz et al. ([HMS06]) and of Li ([Li06].)

the other two modal operators:

(A0) $X_i\varphi \leftrightarrow A_i\varphi \wedge K_i\varphi$.

The versatility of this approach to modeling awareness lies in the fact that we can capture different interpretations of awareness by imposing restrictions on the awareness functions. For example, as it is done in [FH88], one may require (*sub*) that awareness be closed under subformulas so that, if $\varphi \in \mathcal{A}_i(w)$ and ψ is a subformula of φ, then $\psi \in \mathcal{A}_i(w)$ as well. More strongly, we could (*gpp*) build the awareness sets starting from a set of primitive proposition $\Psi \subseteq \Phi$ and stipulating that the sets contain *exactly* those formula that mention only primitive propositions belonging to Ψ. Moreover, (*ka*) one could require that an agent knows *what* formulas she is aware of, so that any formula φ belongs to $\mathcal{A}_i(w)$ if and only if its truth set $\|\varphi\|$ belongs to $\mathcal{N}_i(w)$[10]

Specific syntactic axioms correspond to each of the restrictions listed above:

sub Closure under Subformulas

(A1) $A_i\neg\varphi \rightarrow A_i\varphi$

(A2) $A_i(\varphi \wedge \psi) \rightarrow A_i\varphi \wedge A_i\psi$

(A3) $A_iX_j\varphi \rightarrow A_i\varphi$

(A4) $A_iK_j\varphi \rightarrow A_i\varphi$

(A5) $A_iA_j\varphi \rightarrow A_i\varphi$

gpp Generated by Primitive Propositions

(A6) $A_i\neg\varphi \leftrightarrow A_i\varphi$

(A7) $A_i(\varphi \wedge \psi) \leftrightarrow A_i\varphi \wedge A_i\psi$

(A8) $A_iX_j\varphi \leftrightarrow A_i\varphi$

(A9) $A_iK_j\varphi \leftrightarrow A_i\varphi$

(A10) $A_iA_j\varphi \leftrightarrow A_i\varphi$

ka Knowledge of Awareness

(A11) $A_i\varphi \rightarrow K_iA_i\varphi$

(A12) $\neg A_i\varphi \rightarrow K_i\neg A_i\varphi$

[10] In Kripke models augmented with awareness operators, *ka* would correspond to the semantic condition that, for all pairs of worlds w, v such that $(w, v) \in \mathcal{K}_i$, we have $\mathcal{A}_i(w) = \mathcal{A}_i(v)$.

1.3 Impossible Possible Worlds Structures

The approach to dealing with logical omniscience based on Kripke structures supplemented with impossible possible worlds relaxes the assumption that possible worlds be logically consistent[11]. Thus, for example, at a particular world w, the agent might know φ, and still not know ψ even if ψ is a logical consequence of φ. This happens when, at w, the agent considers as possible a world w^* in which φ holds while ψ does not. Or, at w, the agent may consider possible a world w^* in which neither φ nor $\neg\varphi$ are true; hence, the agent, at w does not know φ nor does she know $\neg\varphi$.

The same approach can be used to relax the logical omniscience properties of neighborhood semantics: recall that axiom *RE* holds in all neighborhoods frames because, in general, $\|\varphi\|^M = \|\psi\|^M$ whenever φ and ψ are logically equivalent. But this need not be the case if we admit impossible worlds in the construction. In particular, there could be an impossible world w^* such that $(M, w^*) \models \varphi$ but $(M, w^*) \not\models \psi$. In this model, the intensions of φ and ψ differ even if the two formulas are logically equivalent. Hence $K_i\varphi \leftrightarrow K_i\psi$ need not follow from $\varphi \leftrightarrow \psi$. This also shows that, in the system **EM**, interpreted over impossible worlds, agents' knowledge[12] need not be closed under logical consequence.

Formally, we consider language $\mathcal{L}$, and we define an impossible worlds structure to be a tuple $M^I = (W, W^*, \mathcal{N}_1^*, \ldots, \mathcal{N}_n^*, \pi, \tau)$ where

W is a non-empty set of possible worlds;

W^* is a non-empty set of impossible worlds;

$\mathcal{N}_i^* : W \to 2^{2^{W \cup W^*}}$ are n neighborhood functions;

$\pi : W \times \Phi \to \{0, 1\}$ is a valuation function;

$\tau : W^* \times \Psi \to \{0, 1\}$, is a valuation function that, in each impossible world, assigns a truth value to some subset Ψ of the set of all formulas in $\mathcal{L}$.

We call the tuple $(W, W^*, \mathcal{N}_1^*, \ldots, \mathcal{N}_n^*, \tau)$ an impossible worlds frame. The satisfiability relation behaves standardly on possible worlds belonging to W, whereas the truth assignment in the impossible worlds belonging to W^* is arbitrary, and yielded by the syntactic valuation τ:

$$(M, w^*) \models \varphi \text{ iff } \tau(w^*, \varphi) = 1, \text{ where } w^* \in W^*$$

[11] For early treatments of the idea, cf. [Cre73], [Kri65] and [RB79].

[12] Notice that in the IPW context we are *not* distinguishing between implicit and explicit knowledge, and the agents are not logical omniscient with respect to the operators K_i.

The notions of validity in a model, validity in a frame, and logical consequence are defined with respect to *possible* worlds only, while, of course the truth set of a formula φ is now defined over $W \cup W^*$.

The agents are not logical omniscient. Consider functions $\mathcal{N})^*_i$, the restrictions of $\mathcal{N}^*_i$ to W. We can have that, at an impossible world w^*, $(M, w^*) \models \varphi$ but $(M, w^*) \not\models \psi$ even when $\varphi \leftrightarrow \psi$ is valid, so that RE is not valid except that in the case in which the domain of neighborhood functions is restricted to the possible worlds belonging to W. Even if impossible worlds are structured in such a way that RE is valid, it need not be the case that, in the system **EM**, agents's knowledge is closed under logical consequence. Indeed, even if the structure is supplemented, i.e. has property (m), $K_i(\varphi \wedge \psi) \rightarrow K_i\psi$ need not hold. This happens when, for instance, there is an impossible world w^* such that $(M, w^*) \models \varphi \wedge \psi$ yet, say, $(M, w^*) \not\models \psi$. In this case, we have that $w^* \in \|\varphi \wedge \psi\|$, although $w^* \notin \|\psi\|$. Hence, (m) is not sufficient to ensure that $\|\psi\| \in \mathcal{N}^*_i(w)$. Moreover, if the system *contains the unit* (i.e. axiom N holds, stating that each neighborhood contains the whole space, which in this case translates into $\{W \cup W^*\} \in \mathcal{N}_i$), it can be the case that, for *some* tautology $\top$ and impossible world W^*, $(M, w^*) \not\models \top$, hence $\|\top\| \subsetneq \{W \cup W^*\}$, and $\|\top\| \notin \mathcal{N}^*_i(w)$ even if $\{W \cup W^*\} \in \mathcal{N}_i(w)$.

Building on the observation relative to the failures of logical omniscience listed above, it is possible to show that, by imposing the appropriate conditions on the structure of impossible worlds, we can restore some aspects of the agents's omniscience, that is to say, we can can validate specific axioms of classical epistemic logic[13]. In particular, for all φ, ψ belonging to $\mathcal{L}$ and w, w^* belonging to W, W^*, respectively, consider the (largest) class of valuations

$\mathcal{T}_{RE}$ such that, for all $\tau \in \mathcal{T}_{RE}$, if $(M, w) \models \varphi \leftrightarrow \psi$, then $w^* \in X \subseteq \mathcal{N}^*_i(w)$ implies that $\tau(w^*, \varphi) = \tau(w^*, \psi)$;

$\mathcal{T}_M$ such that, for all $\tau \in \mathcal{T}_M$, if $\tau((w^*, \varphi \wedge \psi) = 1$, then $\tau(w^*, \varphi) = \tau(w^*, \psi,) = 1$;

$\mathcal{T}_N$ such that, for all $\tau \in \mathcal{T}_N$, $\tau(w^*, \top) = 1$.

It should be clear that, failing the counterexamples to the agents's omniscience above, the class of all impossible world frames supplemented with $\tau \in \mathcal{T}_{RE}$ validates axiom RE; the class of all impossible world frames satisfying (m) and supplemented with $\tau \in \mathcal{T}_M$, validates axiom M; the class of all impossible world frames satisfying (n) and supplemented with $\tau \in \mathcal{T}_N$, validates axiom N.

[13] Cf. [Wan90].

1.4 Comparison

The equi-expressivity between AWA and IPW interpreted over Kripke semantics is studied in [Wan90], [Thi93], [FHMV95] and [HP07]. In this section, I show that the equi-expressivity still holds if AWA and IPW are interpreted over neighborhood semantics. Note that the following proposition holds for *general* awareness, i.e. systems where no restrictions are imposed over the awareness operators.

Proposition 1.1. Let $M^A = (W, \mathcal{N}_1, \ldots, \mathcal{N}_n, \mathcal{A}_1, \ldots, \mathcal{A}_n, \pi)$ be an awareness structure and $M^I = (W, W^*, \mathcal{N}_1^*, \ldots, \mathcal{N}_n^*, \pi, \tau)$ be an impossible possible worlds structure. It is possible to define M^A and M^I in such a way that, for any world $w \in W$, it holds that $(M^A, w) \models \varphi$ iff $(M^I, w) \models \varphi'$, where φ' consists of the formula φ in which every instance of X_i is replaced with K_i, and vice versa.

Proof. [⇒]. Let M^A be given. Construct M^I as follows:

- The set W of possible worlds and the valuation function π defined on them are the same as in M^A.
- The set of impossible worlds W^* is yielded by the awareness sets of the agents at each world: $W^* = \{w_i^* : i = 1, \ldots, n\}$.
- The assignment τ is such that $\tau(w_i^*, \varphi) = 1$ iff $(M^A, w) \models A_i\varphi$.
- The neighborhood functions $\mathcal{N}_i^* : W \to 2^{2^{W \cup W^*}}$ are the extension of the $\mathcal{N}_i$'s to the set $W \cup W^*$ such that if $X = \|\varphi\|^{M^A} \subseteq \mathcal{N}_i(w)$, then $X^* = X \cup \{w_i^*\} \cup \{v_j^* : (M, v_j^*) \models \varphi$, for all $v \in W, j = 1, \ldots, n\}$ and $X^* \subseteq \mathcal{N}_i^*(w)$.

Of course, the truth set of a formula φ is now defined with respect to both standard and impossible worlds: $\|\varphi\|^{M^I} = \{w \in W \cup W^* : (M, w) \models \varphi\}$. Since the satisfiablity relation for M^I behaves standardly on standard worlds, if φ is one of the atoms, or any well formed combination of atoms and boolean connectives, then obviously $(M^A, w) \models \varphi$ iff $(M^I, w) \models \varphi'$. If φ is $X_i\psi$, we show that (i) if $(M^A, w) \models X_i\psi$, then $(M^I, w) \models K_i\psi$, and that (ii) if $(M^A, w) \not\models X_i\psi$, then $(M^I, w) \not\models K_i\psi$. *Ad* (i): if $(M^A, w) \models X_i\psi$, then $\|\psi\|^{M^A} \in \mathcal{N}_i(w)$ and $\psi \in \mathcal{A}_i(w)$, because $(M^A, w) \models K_i\psi$ and $(M^A, w) \models A_i\psi$, respectively. By construction, $X^* = \|\psi\|^{M^A} \cup \{w_i^*\} \cup \{v_j^* : (M^I, v_j^*) \models \psi$; for all $v \in W, j = 1, \ldots, n\}$. Thus, if $(M^I, w_i^*) \models \psi$, then $X^* = \|\psi\|^{M^I}$, as desired. But, by construction, $(M^I, v_i^*) \models \psi$, since $\psi \in \mathcal{A}_i(w)$. *Ad* (ii): If, one one hand, $(M^A, w) \not\models X_i\psi$ because $(M^A, w) \not\models K_i\psi$, then $\|\psi\|^{M^A} \not\subseteq \mathcal{N}_i(w)$ and *a fortiori* $\|\psi\|^{M^I} \not\subseteq \mathcal{N}_i^*(w)$ and $(M^I, w) \not\models K_i\psi$. If, on the other hand, $(M^A, w) \not\models X_i\psi$ because $(M^A, w) \not\models A_i\psi$, then $(M^I, w_i^*) \not\models \psi$ and $X^* \neq \|\psi\|^{M^I}$, or $\|\psi\|^{M^I} \not\subseteq \mathcal{N}_i^*(w)$, hence $(M^I, w) \not\models K_i\psi$.

[$\Leftarrow$]. Let M^I be given. Construct M^A as follows: the set of possible worlds W is the same as in M^I; the neighborhoods $\mathcal{N}_i$ are the restriction to W of the neighborhoods $\mathcal{N}_i^*$; the awareness set for agent i at w is given by the formulas which are in the scope of the knowledge operator in the corresponding IPW state, that is $\mathcal{A}_i(w) = \{\varphi : (M^I, w) \models K_i\varphi\}$; the valuation π agrees with the valuation in M^I. Again, the only less obvious case is for the modalities K_i. Thus, let φ be $K_i\psi$. By construction, $(M^I, w) \models K_i\psi$ iff $\|\psi\|^{M^I} \in \mathcal{N}_i^*(w)$, which implies $\|\psi\|^{M^A} \in \mathcal{N}_i(w)$; hence $(M^A, w) \models K_i\psi$. Moreover if $(M^I, w) \models K_i\psi$, then, by construction, $(M^A, w) \models A_i\psi$. By the semantics of X_i, we have that $(M^A, w) \models X_i\psi$, as desired. To complete the proof, notice that if $(M^I, w) \not\models \psi$, then $\psi \notin \mathcal{A}_i(w)$, hence $(M^A, w) \not\models K_i\psi$. ■

We can also find a correspondence between, on the one hand, restrictions on the construction of awareness sets in awareness structures and, on the other, analogous restrictions on the definition of the valuation function τ in impossible worlds structures. Consider for instance *gpp* awareness. We stipulate the following restrictions on the behavior of τ:

(i)	$\tau(w^{\mathcal{A}_i(w)}, \varphi) = 1$	iff	$\tau(w^{\mathcal{A}_i(w)}, \neg\varphi) = 1$
(ii)	$\tau(w^{\mathcal{A}_i(w)}, \varphi \wedge \psi) = 1$	iff	$\tau(w^{\mathcal{A}_i(w)}, \varphi) = 1$ and $\tau(w^{\mathcal{A}_i(w)}, \psi) = 1$
(iii)	$\tau(w^{\mathcal{A}_i(w)}, K_i\varphi) = 1$	iff	$\tau(w^{\mathcal{A}_i(w)}, \varphi) = 1$.

From these semantic conditions it easily follows that

Proposition 1.2. Let the structures M^A and M^I be as in the proposition above, and let awareness be generated by primitive propositions. Then $\tau(w^{\mathcal{A}_i(w)}, \varphi) = 1$ if and only if $\tau(w^{\mathcal{A}_i(w)}, p) = 1$ for all atoms p occurring in φ.

Proof. A straightforward induction on φ. If φ is the primitive proposition p, then the claim holds obviously. If φ has one of the forms $\neg\psi$, $\psi' \wedge \psi''$, $K_i\psi$, then the induction hypothesis and conditions (i), (ii) and (iii), respectively, prove the claim and complete the induction. ■

Thus, the impossible worlds in the construction behave as the awareness sets generated by primitive propositions do. More precisely, for all $\varphi \in \mathcal{L}$ and w, w^* belonging to W, W^* respectively, consider the (largest) class of valuations

$\mathcal{T}_{gpp}$ such that, for all $\tau \in \mathcal{T}_{gpp}$, conditions (i)-(iii) are satisfied.

The class of all impossible world frames supplemented with $\tau \in \mathcal{T}_{gpp}$ satisfies the construction of proposition 1.1 when awareness is generated by primitive propositions.

2 Predicate Systems

The results obtained in the first section at the propositional level are here proven to hold at the predicate level as well.

2.1 Quantified Logic of Awareness

One of the motivations behind the introduction, in [Sil06], of predicate awareness logics, stems from the inadequate expressivity of the propositional logics of awareness in which "awareness of unawareness" cannot be expressed. Instances of awareness of one's own unawareness are abundant (for example, an agent knows that there exists a prime larger than the largest known prime, although she does not know what number that is), but a propositional characterization, like $A_i \neg A_i \varphi$, fails when awareness is understood as generated by primitive propositions. Such an interpretation of awareness is prominent in the economics literature[14], hence the need to overcome the limitation in its expressive power. Halpern and Rêgo explore one possible solution (propositional quantifiers) in [HR06], while [Sil06] explores a different one (predicate awareness logics). The latter approach is recalled here, and then formally comapred with quantified IPW structures. I now define first-order classical epistemic systems, and then extend them to incorporate awareness and explicit knowledge operators.

To the language $\mathcal{L}$, we add a countable set of n-ary predicates $P, Q, R, \ldots$ for any $n \geq 1$, a countable set of variables $\mathcal{V}$ and the universal quantifier $\forall$. Call $\mathcal{L}^Q$ the language thus obtained. The expression $\varphi(x)$ denotes that x occurs free in φ, while $\varphi[y/x]$ stands for the formula φ in which the free variable x is replaced with the free variable y. An *atomic formula* has the form $P(x_1, \ldots, x_n)$, where P is a predicate symbol of arity n. If **S** is a classical propositional modal logic, **QS** is given by the following axioms:

S All the axioms of **S**

$\forall$ $\forall x \varphi(x) \rightarrow \varphi[y/x]$

Gen From $\varphi \rightarrow \psi$, infer $\varphi \rightarrow \forall x \psi$, where x is not free in φ

[14] As it is formally shown in [Hal01] and [HR05], the approach to awareness in the economics literature (cf. [MR99], [HMS06]) can be seen as the system of Fagin and Halpern's logic of awareness [FH88] in which awareness is generated by primitive propositions and agents have knowledge of their own awareness, i.e. the case in which the awareness operators satisfy axioms (A6)-(A12) above.

As to the semantics, a *constant domain neighborhood frame* is a tuple $\mathcal{F} = (W, \mathcal{N}_1, \ldots, \mathcal{N}_n, D)$, where W is a set of possible worlds, D is a non-empty set called the domain, and each $\mathcal{N}_i$ is a *neighborhood* function from W to 2^{2^W}. A *model* based of a frame $\mathcal{F}$ is a tuple $M^Q = (W, \mathcal{N}_1, \ldots, \mathcal{N}_n, D, I)$, where I is a classical first-order interpretation function. A *substitution* is a function $\sigma : \mathcal{V} \to D$. If a substitution σ' agrees with σ on every variable except x, it is called an x-variant of σ, and such a fact is denoted by the expression $\sigma \sim_x \sigma'$. The satisfiability relation is defined at each state relative to a substitution σ:

$(M, w) \models_\sigma P(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in I(P, w)$ for each n-ary predicate symbol P.

$(M, w) \models_\sigma \neg\varphi$ iff $(M, w) \not\models_\sigma \varphi$

$(M, w) \models_\sigma \varphi \wedge \psi$ iff $(M, w) \models_\sigma \varphi$ and $(M, w) \models_\sigma \psi$

$(M, w) \models_\sigma K_i\varphi$ iff $\{v : (M, v) \models_\sigma \varphi\} \in \mathcal{N}_i(w)$

$(M, w) \models_\sigma \forall x \varphi(x)$ iff for each $\sigma' \sim_x \sigma, (M, w) \models_{\sigma'} \varphi(x)$

We now extend the language $\mathcal{L}^Q$ to the language $\mathcal{L}^{QA}$ by adding the modalities A_i and X_i. A first-order awareness model (with arbitrary awareness sets) is the a tuple $M^{QA} = (W, \mathcal{N}_1, \ldots, \mathcal{N}_n, \mathcal{A}_1, \ldots, \mathcal{A}_n, I, D)$, which supplements a first-order neighborhood model M^Q with n awareness sets $\mathcal{A}_1, \ldots, \mathcal{A}_n$. The semantic clauses for the new operators are the straightforward

$(M, w) \models_\sigma A_i\varphi$ iff $\varphi \in \mathcal{A}_i(w)$

$(M, w) \models_\sigma X_i\varphi$ iff $(M, w) \models_\sigma K_i\varphi$ and $(M, w) \models_\sigma A_i\varphi$.

2.2 Awareness and Quantification

In a fashion similar to the interpretation of awareness as generated by primitive propositions in the propositional case, we can interpret awareness in a first order system as being *generated by atomic formulas* (*gaf*), in the sense that i is aware of φ at w iff i is aware of all atomic subformulas in φ. Thus, for each i and w, there is a set (call it *atomic awareness set* and denote it $\Phi_i(w)$) such that $\varphi \in \mathcal{A}_i(w)$ iff φ mentions only atoms appearing in $\Phi_i(w)$. This interpretation of awareness can be captured axiomatically. The axioms relative to the boolean and modal connectives are the usual ones (see axioms (A6)-(A10) above.)

The introduction of the axioms concerning quantifiers needs a further preliminary discussion. Note that in the first-order set-up we can have a more fine-grained definition of the atomic awareness sets than we can in the propositional case. In particular, rather than constructing the atomic awareness sets as unstructured lists of atoms (as it must be the case for propositional awareness) we can now have them generated by the semantic structure itself. The idea (akin in spirit to the distinction between an *inner* and an *outer* domain used in certain models with varying domains[15] or in free logics) is to distinguish, for each agent i and each world w a *subjective* domain $D_i(w) \subseteq D$ and impose that (i) if $P(x_1, \ldots, x_n) \in \Phi_i(w)$, then $x_i \in D_i(w)$. Intuitively, the values of the functions D_i represent the objects in D of which agent i is aware at w. Moreover, a further source of unawareness may lie in the fact that the agent lacks awareness of a predicate. To formalize this, we introduce, for each i, a subjective interpretation function I_i that agrees with I except that, possibly, assigns a smaller extension to some predicates P. We can then say that (ii) if $P(x_1, \ldots x_n) \in \Phi_i(w)$, then $\langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in I_i(P, w)$. We now stipulate the following

Definition 2.1. The atomic awareness set functions Φ_i from W to the set of atoms of $\mathcal{L}^{QA}$ are defined in such a way that $P(x_1, \ldots x_n) \in \Phi_i(w)$ iff conditions

(i) if $P(x_1, \ldots, x_n) \in \Phi_i(w)$, then $x_i \in D_i(w)$ and

(ii) if $P(x_1, \ldots x_n) \in \Phi_i(w)$, then $\langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in I_i(P, w)$

hold.

We also need to introduce a family of special n-ary predicates[16] $A!_i$ whose intuitive meaning is "i is aware of objects $\sigma(x_1), \ldots, \sigma(x_n)$. More formally, a first-order structure for weakly-*gaf* awareness is a tuple $M^{QA} = (W, \overrightarrow{\mathcal{N}_i}, \overrightarrow{\mathcal{A}_i}, I, \overrightarrow{I_i}, D, \overrightarrow{D_i})$, where $(W, \overrightarrow{\mathcal{N}_i})$ is a first-order classical frame; $\overrightarrow{\mathcal{A}_i}$ are n awareness functions associating a set of formulas to each agent and world; D and I are the (constant) domain and a (standard) first-order interpretation function, respectively; $D_i : \mathcal{V} \to D$ is a subjective domain function agreeing (possibly partially) with D ; $I_i : (P^n, w) \to D_i^n$ is a subjective intepretation function which, relative to D_i agrees (possibly partially) with I. As to the awareness predicate $A_i!$, we impose that $(M, w) \models_\sigma A!_i(x)$ iff $\sigma(x) \in D_i(x)$.

We are now ready to state the axioms regulating the behavior of the awareness operators with respect to quantifiers. First, note that if we were to fully close the awareness operators under the existential quantifier, we

[15] See for example [HC96], ch. 14.

[16] Such predicates are akin to the existence predicate in free logic. However, the awareness system considered here is not based on free logic: the special awareness predicates will only be used to limit the range of possible substitutions for universal quantifiers *within* the scope of awareness operators. The behavior of quantifiers is otherwise standard.

would make an important reason behind the introduction of quantified logic of awareness moot, since $A_i\exists x\varphi(x) \to A_i\varphi[y/x]$ precludes the possibility of being "aware of unawareness." For this reason[17] we only require the weak existential closure

$$A\exists \quad A_i\varphi[y/x] \to A_i\exists x\varphi(x).$$

With regard to the universal quantifier, the weak closure of awareness with respect to it needs to be qualified, since we can only sensibly substitute with objects from the subjective domain of quantification. Thus,

$$A\forall \quad A_i\forall x\varphi(x) \to (A!_i(y) \to A_i\varphi[y/x]).$$

We say that awareness is *weakly* generated by atomic formulas if axioms (A6)-(A10) and axioms $A\exists$ and $A\forall$ above hold.

2.3 Quantified Impossible Worlds and Awareness

Let us now turn our attention to *quantified impossible possible worlds structures.* The system defined below is based on the one introduced in [Ran82b]. In order to make the comparison with quantified awareness, some characteristics of Rantala's IPW structures had to be adjusted. In particular, while Rantala uses a language containing constant terms, we restrict ourselves here to the a language containing only individual variables. Semantically, as throughout this article, we interpret the language over neighborhood structures rather than Kripke structures.

The system is based on the language $\mathcal{L}^Q$ obtained from $\mathcal{L}^{QA}$ by dropping the modal operators A_i and X_i. A first-order IPW structure is a tuple $M^{QI} = (W, W^*, \mathcal{N}_1^*, \ldots, \mathcal{N}_n^*, I, \tau, D)$, where W and W^* are non-empty sets containing possible and impossible worlds, respectively. As usual, we have a function $\sigma : \mathcal{V} \to D$ assigning to each variable in $\mathcal{L}^{QI}$ an individual from the domain D. The neighborhood functions $\mathcal{N}_i^* : W \to 2^{2^{W\cup W^*}}$ assign to each world, for each agent, a set of subsets of $W \cup W^*$. The classic first-order interpretation function I ranges over $P^n \times W$ only (where P^n is any n-ary predicate, for any n) and has D^n as its domain. The satisfiability relation is defined recursively based on I in a standard way. To give a truth value to formulas at $w^* \in W^*$, we use the syntactic assignment $\tau : W^* \times \Psi \to \{0, 1\}$ that assigns, in each impossible world, a truth value to some subset of formulas of $\mathcal{L}^{QI}$. We finally set that, for impossible worlds,

$$(M, w^*) \models_\sigma \varphi \text{ iff } \tau(w^*, \varphi) = 1$$

[17] This observation is made also in [HR06], where analogous conclusions are drawn from it.

Mutatis mutandis, proving the equi-expressivity of the awareness and impossible possible worlds approaches in the predicate case does not present particular differences from the propositional case and the proof of Proposition 1.1 carries over easily to the predicate case, as the following proposition shows:

Proposition 2.2. Let $M^{QA} = (W, \mathcal{N}_1, \ldots, \mathcal{N}_n, \mathcal{A}_1, \ldots, \mathcal{A}_n, I, D)$ be an awareness structure and $M^{QI} = (W, W^*, \mathcal{N}_1^*, \ldots, \mathcal{N}_n^*, \tau, I, D)$ be an impossible possible worlds structure. Fix an assignment σ common to both M^{QA} and M^{QI}. It is possible to define M^{QA} and M^{QI} in such a way that, for any world $w \in W$, it holds that $(M^A, w) \models_\sigma \varphi$ iff $(M^I, w) \models_\sigma \varphi'$, where φ' consists of the formula φ in which every instance of X_i is replaced with K_i, and vice versa.

Proof. For the direction from left to right, the model M^{QI} has elements W, I, and D in common with M^{QA}. The neighborhoods $\mathcal{N}_i^*(w)$ are constructed as they are in proposition 1.1, and so is the assignment τ. The argument from proposition 1.1 carries over obviously. For the direction from right to left, keep W, D, and I constant and construct M^{QI} along the lines of the construction in the proof of proposition 1.1. ∎

While the previous proposition shows that the equi-expressivity result of proposition 1.1 carries straightforwardly over to the predicate case when dealing with *general* awareness operators, it is not obvious that we can identify a class of valuations corresponding to *weakly generated by atomic formulas* awareness. It turns out that we can define a such a class.

First we need to define $A!_i$ predicates corresponding to the ones we defined on awareness structures. We set that $(M^{QI}, w) \models_\sigma A!_i y$ iff $y \in D_i(w)$.

To ease readability, the propositional restrictions (i)-(iii) of page 16 are restated here:

(i)	$\tau(w^{\mathcal{A}_i(w)}, \varphi) = 1$	iff	$\tau(w^{\mathcal{A}_i(w)}, \neg\varphi) = 1$
(ii)	$\tau(w^{\mathcal{A}_i(w)}, \varphi \wedge \psi) = 1$	iff	$\tau(w^{\mathcal{A}_i(w)}, \varphi) = 1$ and $\tau(w^{\mathcal{A}_i(w)}, \psi) = 1$
(iii)	$\tau(w^{\mathcal{A}_i(w)}, K_i\varphi) = 1$	iff	$\tau(w^{\mathcal{A}_i(w)}, \varphi) = 1$.

After adding the conditions

(iv) $\tau(w^{\mathcal{A}_i(w)}, \varphi[y/x]) = 1$ implies $\tau(w^{\mathcal{A}_i(w)}, \exists x\varphi(x)) = 1$, and
(v) $\tau(w^{\mathcal{A}_i(w)}, \forall x\varphi(x)) = 1$ implies that if $(M^{QI}, w) \models_\sigma A!_i y$ then $\tau(w^{\mathcal{A}_i(w)}, \varphi(y)) = 1$,

we can finally state the following

Proposition 2.3. Let $M^{QA}_{wgaf} = (W, \overrightarrow{\mathcal{N}_i}, \overrightarrow{\mathcal{A}_i}, \overrightarrow{I_i}, \overrightarrow{D_i}, I, D)$ and let M^{QI}_{wgaf} be $(W, \overrightarrow{\mathcal{N}^*_i}, \tau, \overrightarrow{I_i}, \overrightarrow{D_i}, I, D)$. Let awareness be weakly generated by atomic formulas, the functions Φ_i be defined as in definition 2.1, and let $\overrightarrow{I_i}, \overrightarrow{D_i}, I$, and D agree across the two models. Then

$$\tau(w^{\mathcal{A}_i(w)}, P(x_1, \ldots, x_n)) = 1 \text{ iff } P(x_1, \ldots, x_n) \in \Phi_i(w),$$

and axioms (A6), (A7), (A10), (A∃), (A∀) hold iff conditions (i)-(v) hold, respectively.

Proof. We only need to show that the correspondence holds between axioms (A∃), (A∀) and conditions (iv) and (v), since the previous case were already considered in proposition 1.2. Recall that $\tau(w^{\mathcal{A}_i(w)}, \varphi) = 1$ iff $(M^{QA}_{wgaf}, w) \models_\sigma A_i\varphi$. It is then straightforward to see that (A∃) implies (iv) (and vice versa) and that (A∀) implies (v) (and vice versa.) ■

Thus, the impossible worlds in the construction behave as the awareness sets weakly generated by atomic formulas do. More precisely, for all $\varphi \in \mathcal{L}$ and w, w^* belonging to W, W^* respectively, consider the (largest) class of valuations

$\mathcal{T}_{wgaf}$ such that, for all $\tau \in \mathcal{T}_{wgaf}$, conditions (i)-(v) are satisfied.

The class of all impossible world frames supplemented with $\tau \in \mathcal{T}_{wgaf}$ satisfies the construction of proposition 2.2 when awareness is waekly generated by atomic formulas.

3 Conclusions

We have established that, both at the propositional and the predicate level, pairing neighborhood semantics and either awareness or impossible worlds structures limits the incidence of the logical omniscience problem.

Neighborhood semantics is crucial to important applications[18] and possess interesting properties[19] which can result important in attacking problems as, for example, interpreting modalities as high probability operators[20]. Although neighborhood semantics reduces the logical omniscience of agents if compared with standard Kripke semantics, it does not entirely eliminate

[18] For instance, Parikh's *game logic* is interpreted over non-normal structures; also Pauly's *coalition logic* is also based on neighborhood semantics. Neighborhood semantics seems to appeal to models of *social software*, cf. [Par01].

[19] For studies on predicate systems based on neighborhood semantics cf. for instance [AC02]—where interesting facts about the relation between neighborhood semantics and the Barcan formulas are proven—and [ACP06] where a general completeness proof is provided.

[20] Cf. again [ACP06].

it. In fact, applications resorting to classical systems as **EM** do experience the problem of the omniscience of agents with respect to logical consequence. Thus, an approach that limits the incidence of logical omniscience in classical systems of epistemic logic could result desirable and useful. I have explored the idea of pairing neighborhood and awareness structures—resulting, to the best of my knowledge, in a novel epistemic system—in [Sil07], proving the decidability of an expressive fragment of quantified logics of awareness interpreted over neighborhood structures.

In this paper I carry on this line of research by comparing awareness logics interpreted over neighborhood structures and epistemic logic interpreted over impossible worlds structures, both at the propositional and at the predicate level. The main results indicate that the awareness and the impossible worlds approaches are of equal expressive power, and suggest (as analogous results that are known to hold in the case of Kripke systems do[21]) that the choice of one formalization over another should be based on pragmatical, rather than theoretical considerations. In order to carry on the comparison, I needed to define neighborhood structures based on impossible worlds. That construction constitutes another contribution of this paper.

Future research on this topic includes extensions of systems of quantified logic of awareness with group-knowledge operators—in particular, extensions with the fix-point, "common knolwedge" operator. Given that neighborhood semantics seem to find natural applications in systems of *social software*, the introduction of group epistemic operators is desirable. The importance of common knowledge for a Lewisian account of social convention or social norms[22] is stressed in [CS03], [Sil05] and [Sil08]. In [Sil08] is also pointed out that the formalization of Lewis' most general account of convention requires the full expressive power of predicate logic. Thus, quantified logic of awareness with common knowledge operators seems a natural and relevant extension for the system studied here.

[21] Cf. for instance [HP07].
[22] For the former, cf. [Lew69], for the latter, [Bic06].

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