A note on “The No Alternatives Argument” by Richard Dawid, Stephan Hartmann and Jan Sprenger*

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Abstract

The defense of *The No Alternatives Argument* in a recent paper by R. Dawid, S. Hartmann and J. Sprenger (Preprint, dated February 14, 2012, URL: http://philsci-archive.pitt.edu/9038/) rests on the assumption (among others) that the number of acceptable alternatives to a scientific hypothesis is independent of the complexity of the scientific problem. This note proves a slight generalisation of the main theorem by Dawid, Hartmann and Sprenger, where this independence assumption is no longer necessary. In passing, some of the other assumptions are also discussed.

1 Introduction

The *No-Alternatives Argument* is the thesis that a given hypothesis H can be confirmed, in the sense of Bayesian confirmation theory, by the event that the scientific community has so far failed to find an acceptable alternative to H. In a remarkable paper [2] circulated in early 2012, Richard Dawid, Stephan Hartmann and Jan Sprenger provide a formal Bayesian analysis of the No-Alternatives Argument (for an invitation to Bayesian epistemology, cf. e.g. Bovens and Hartmann [1]).

Their formal framework is based on a probability space consisting of a sample space (set of possible worlds) Ω and a (countably additive) probability

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measure P on some σ-algebra of subsets of Ω, such that the propositions "The scientific community has not yet found an alternative to H" (henceforth denoted F_A) and "The hypothesis H is empirically adequate" (henceforth denoted T) as well as the functions

- $Y : \Omega \to \mathbf{N}$, where $Y(\omega)$ denotes the number of alternatives to H in world ω,
- $D : \Omega \to \mathbf{N}$, where $D(\omega)$ denotes the degree of difficulty (assumed to be a nonnegative integer), in world ω, of the scientific problem addressed by hypothesis H,

are all P-measurable. (Herein, $\mathbf{N}$ denotes the set of nonnegative integers.)

In this framework, Richard Dawid, Stephan Hartmann and Jan Sprenger rigorously prove that the No-Alternatives Argument (i.e. $P(T|F_A) > P(T)$ in the above notation) holds under the following assumptions:

A0. Y and D are independent.

A1. T and F_A are conditionally independent given Y: $P(T \cap F_A|\{Y = n\}) = P(T|\{Y = n\})P(F_A|\{Y = n\})$ for every $n \in \mathbf{N}$.

A2. $P\{Y = n\} < 1$ for all $n \in \mathbf{N}$.

A3. $P(F_A|\{Y = k\} \cap \{D = j\})$ is decrasing in k for every fixed j as well as increasing in j for every fixed k.

A4. $P(T|\{Y = k\})$ is decrasing in k.

A5. There exist $i, j \in \mathbf{N}$ such that

1. $j > i$,
2. $P\{Y = i\}, P\{Y = j\} > 0$,
3. $P(F_A|\{Y = i\} \cap \{D = k\}) > P(F_A|\{Y = j\} \cap \{D = k\})$ for some $k \in \mathbf{N}$,
4. $P(T|\{Y = i\}) > P(T|\{Y = j\})$.

Assumption A2 is, strictly speaking, redundant, as it follows from A5. Informal motivations of all assumptions can already be found in the paper by Richard Dawid, Stephan Hartmann and Jan Sprenger.

This short note shows how to generalise this astonishing theorem by replacing A3 and A5 with two stronger assumptions (A3' and A5'). At the outset, however, we shall give a more formal motivation of A5 than the one offered by Dawid, Hartmann and Sprenger; this will later be useful in motivating the modified assumption A5'.

2 Motivating A5 based on A3

In order to give a more formal vindication of assumption A5 than the one provided by Dawid, Hartmann and Sprenger, one might consider giving the following argument, which operates on a *reductio ad absurdum* of a denial of A5. Essentially, anyone who rejects A5 either has to reject assumption A3 (for which an independent motivation can be given, cf. Dawid, Hartmann and Sprenger [2, p. 9]) or will have to commit herself to borderline cases.

For notational economy, let us introduce the following abbreviations (the same as in the paper by Dawid, Hartmann and Sprenger):

$$\begin{aligned} y_i &:= P\{Y = i\} \\ t_i &:= P(T|\{Y = i\}) \\ f_{ik} &:= P(F_A|\{Y = i\} \cap \{D = k\}) \end{aligned}$$

With this notation, denying A5 means: For all $j > i \in \mathbf{N}$ such that both $y_i, y_j > 0$ and $t_i > t_j$, one has $f_{ik} \leq f_{jk}$ for all $k \in \mathbf{N}$ — and thus $f_{1k} \leq f_{jk}$ for all $k \in \mathbf{N}$. But since f_{jk} was assumed to be decreasing in j (in A3), this can only mean that for any such j, $f_{jk} \leq f_{\ell k} \leq f_{1k} \leq f_{jk}$ — thus $f_{\ell k} = f_{1k}$ — for all $\ell \leq j$. Hence, for every $j \in \mathbf{N}$ with $y_j > 0$ for which there exists some $i < j$ such that both $y_i > 0$ and $t_i > t_j$, one has $f_{\ell k} = f_{1k}$ for all $\ell \leq j$. Combining this result with the assumption (A4) that t_j is decreasing in j, we conclude that (i) if there are infinitely many $j \in \mathbf{N}$ with $y_j > 0$, then either (iα) f_{jk} will be constant in j for all $k \in \mathbf{N}$ (in case even $t_j < t_i$ for some $i < j$ with $y_i > 0$ for infinitely many j satisfying $y_j > 0$) or (iβ) t_i will be constant in i from the least i satisfying $y_i > 0$ onwards (in case the set of j satisfying both $y_j > 0$ as well as $t_j < t_i$ for some $i < j$ with $y_i > 0$ is merely finite), and (ii) if there are only finitely many $j \in \mathbf{N}$ with $y_j > 0$, then either (iiα) there exists some $i < j$ such that both $y_i, y_j > 0$ and $t_i > t_j$, whence $f_{\ell k} = f_{1k}$ for all ℓ up to j and thus $f_{\ell k}$ is constant in ℓ for all relevant ℓ and all $k \in \mathbf{N}$) or (iiβ) for any $i < j$ satisfying $y_i, y_j > 0$ one has $t_i = t_j$ and thus t_i is constant in i for all i with $y_i > 0$. In all of these situations, Y has no systematic influence on T or F_A whatsoever. Perhaps there exist certain research problems where such an assumption is satisfied, but these are knife-edge cases.

3 How compelling is A0?

A0 is not mentioned in the paper by Dawid, Hartmann and Sprenger explicitly, but it is (i) implicit in the Bayesian Network [2, p. 9, Figure 2] and (ii) in the assertion "D has no direct influence on Y and T (or vice versa)" [2,

p. 8]. The interpretation of D as the degree of difficulty — a measure (of some aspect) of "the complexity of the problem, the cleverness of the scientists, or the available computational, experimental, and mathematical resources" [2, p. 8] — is important for the motivation of assumption A3: The more difficult the problem, given a fixed number of alternatives, the more likely it is that the scientific community has failed to find acceptable alternatives to H. (In particular, A0 cannot be forced by choosing D as some arbitrary non-trivial element of the orthogonal complement of the linear subspace of the Hilbert space $L^2(P)$ generated by Y.)

Given that Y denotes the number of alternatives deemed *acceptable* by the scientific community and thus depends on human cognitive and even social factors, it is not at all clear that there is no area of scientific research where the difficulty of the problem does not have some influence on Y. Nevertheless, the independence of Y and D might be a good approximation to the truth for many interesting classes of research problems — to which the original analysis by Dawid, Hartmann and Sprenger can then be applied without any further modification. That said, it is surely desirable to have some formal analysis of the No-Alternatives Argument that allows for some interdependence of Y and D.

4 Abandoning A0

If one wishes to jettison A0, one will have to impose other assumptions. We suggest to replace A3 and A5 by the following stronger conditions:

A3'. There exists a sequence $\left(\tilde{d}_j\right)_{j\in\mathbf{N}}$ of nonnegative real numbers as well as an infinite matrix $\left(\tilde{f}_{kj}\right)_{j,k\in\mathbf{N}}$ such that $\tilde{d}_j\tilde{f}_{kj} = P\left(F_A \cap \{D=j\} \mid \{Y=k\}\right)$ and $\tilde{f}_{kj}$ is decrasing in k for every fixed j as well as increasing in j for every fixed k.

A5'. With $\tilde{d}$ and $\tilde{f}$ as in A3': There exist $i, j \in \mathbf{N}$ such that

1. $j > i$,
2. $P\{Y=i\}, P\{Y=j\} > 0$,
3. $\tilde{f}_{ik} > \tilde{f}_{jk}$ for some $k \in \mathbf{N}$,
4. $P(T| \{Y=i\}) > P(T| \{Y=j\})$.

Under these assumptions, one can again give a formal justification of the No-Alternatives Argument (proof in Section 7):

Theorem 1. *Assuming A1, A3', A4 and A5', F_A confirms T.*

5 Motivating the new assumptions

It is clear that the new assumptions proposed in Theorem 1, viz. A3' and A5', are — in the presence of A0 — generalisations of A3 and A5, respectively (proof in Section 7):

Remark 2. *A0 and A3 together imply A3'; A0 and A5 together imply A5'.*

But what other motivation can be given for A3' and A5'? Well, once assumption A3' is justified, one may argue for A5' based on A3' in the same manner as assumption A5 can be motivated on the basis of A3 (as it was done in Section 2, where we showed that A5 can only be denied while maintaining A3 in knife-edge cases). So the main task is to argue for A3'.

Now, in order to motivate A3', one should first observe that

$$\begin{aligned} & P\left(F_A \cap \{D=j\} \mid \{Y=k\}\right) \\ = \; & P\left(\{D=j\}|\{Y=k\}\right) P\left(F_A|\{Y=k\} \cap \{D=j\}\right), \end{aligned}$$

which is decreasing in k (for all fixed j) whenever $P\left(F_A|\{Y=k\} \cap \{D=j\}\right)$ decreases faster in k than $P\left(\{D=j\}|\{Y=k\}\right)$ increases (for every fixed j). However, while one may reasonably expect a significant decrease in the probability of the absence of known acceptable alternatives as the number k of alternatives increases — at least while k is not too large and given a fixed degree of difficulty —, a pronounced increase in the probability of having a certain fixed degree of difficulty would be rather surprising, especially for already large k. Thus, it is rather reasonable to assume that $P\left(F_A \cap \{D=j\} \mid \{Y=k\}\right) = \tilde{d}_j \tilde{f}_{kj}$ is decreasing in k (for all fixed j), and hence so is $\tilde{f}_{kj}$.

Secondly, if it were not possible to choose $\tilde{d}$ and $\tilde{f}$ such that $\tilde{d}_j \tilde{f}_{kj} = P\left(F_A \cap \{D=j\} \mid \{Y=k\}\right)$ holds for all k and $\tilde{f}_{kj}$ is increasing in j for every fixed k, this would mean that even by choosing a rapidly decreasing sequence of positive numbers, it would be impossible to get $\tilde{f}_{kj} = \frac{1}{\tilde{d}_j} P\left(F_A \cap \{D=j\} \mid \{Y=k\}\right)$ increasing in j for all k. Hence, $P\left(F_A \cap \{D=j\} \mid \{Y=k\}\right)$ would have to be rapidly decreasing in j — which means that the distribution of the level of difficulty is mostly concentrated on a relatively small initial segment of the nonnegative integers, given k — and the rate of decrease should even accelerate (without any upper bound on that rate) with increasing k. This may be possible for some areas of scientific inquiry, but it also appears to be an extreme case.

Hence, assumption A3' does have some philosophical plausibility, and based on A3' one can argue for A5' as in Section 2.

6 Discussion

The formal justification of the No-Alternatives Argument by Dawid, Hartmann and Sprenger has introduced formal rigour to an important epistemological discussion which will probably be noted well beyond the philosophical community. There is no question that this is a remarkable achievement. The proof even yields an explicit formula for the degree of confirmation associated with the No-Alternatives Argument (even though the parameters may be difficult to determine, as the authors readily admit).

The formal proof — based on Bayesian network and confirmation theory — relies on the assumption that the number Y of acceptable alternatives to a scientific hypothesis H is independent from a certain measure D of contextual influence that can roughly be described as degree of difficulty. This assumption is certainly appealing, but not beyond any reasonable doubt. We have therefore shown how to modify some of the other assumptions, so as to allow for a generalisation of the result by Dawid, Hartmann and Sprenger that also holds for interdependent Y and D.

While these new assumptions do enjoy a certain degree of plausibility, they are also not utterly compelling. But there are, speculatively, more fundamental issues with the No-Alternatives Argument which might motivate the scientific community to resist the temptation of using it too generously.

First, there is a conceptual difficulty: Via the notion of an 'acceptable' alternative theory, sociological (and human cognitive) factors play a pivotal role in the No-Alternatives Argument. But whenever merely contingent historical and sociological factors may preclude scientific reflection about certain otherwise viable alternatives to a dominating hypothesis H, it becomes absurd to count the absence of such reflection as *evidence* for H.

Secondly, there is a teleological problem: One does not need to be a scientific realist to share in the intuition that what gives science its special epistemic status is its (perhaps idealised) perception as a perpetual, unceasing search for true, objectively warranted beliefs (*scientia*). A wide acceptance of the No-Alternatives Argument as a meta-paradigm would put an end to any such understanding of science — most probably with unwelcome repercussions for the role of science in public discourse.

Thirdly, there is a related ethical issue: If there is such a thing as a 'received ethos of science' or a discernable set of society's expectations from science, it probably demands from a community that believes in some hypothesis H to direct all efforts to finding either additional empirical evidence (including, of course, unsuccessful deliberate attempts at falsification) or theoretical justification (including coherence with other established theories), rather than to sit back and dogmatise.

Be this as it may, with the formalisation of the No-Alternatives Argument due to Dawid, Hartmann and Sprenger, it is now possible to give a precise analysis of the strengths and weaknesses of this thesis and even to give formal criteria for suitable domains of application. For formal epistemology, there may be very interesting years ahead!

7 Proofs

For the proofs, it will be helpful to introduce the following abbrevation: For all $j, k \in \mathbf{N}$,

$$g_{kj} := P\left(F_A \cap \{D = j\} \,|\, \{Y = k\}\right).$$

Proof of Theorem 1. As in the original paper by Dawid, Hartmann and Sprenger, the crucial step in the proof is the simplification of

$$\tilde{\Delta} := P(T \cap F_A) - P(T)P(F_A),$$

which is achieved by writing $P(T), P(F_A), P(T \cap F_A)$ in terms of the y's, t's and g's: Using the countable additivity of P, in combination with assumption A1 and the definitions of the y's, t's and g's, we obtain

$$\begin{aligned}
P(F_A) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P\left(F_A \cap \{D = j\} \cap \{Y = i\}\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} y_i g_{ij} \\
P(T \cap F_A) &= \sum_{i=0}^{\infty} P\left(T \cap F_A \cap \{Y = i\}\right) = \sum_{i=0}^{\infty} P\left(T \cap F_A | \{Y = i\}\right) P\{Y = i\} \\
&= \sum_{i=0}^{\infty} P\left(T | \{Y = i\}\right) P\left(F_A | \{Y = i\}\right) P\{Y = i\} \\
&= \sum_{i=0}^{\infty} t_i P\left(F_A \cap \{Y = i\}\right) \\
&= \sum_{i=0}^{\infty} t_i \sum_{j=0}^{\infty} P\left(F_A \cap \{D = j\} \cap \{Y = i\}\right) = \sum_{i=0}^{\infty} t_i \sum_{j=0}^{\infty} g_{ij} y_i \\
P(T) &= \sum_{k=0}^{\infty} P(T|\{Y = k\}) P\{Y = k\} = \sum_{k=0}^{\infty} t_k y_k
\end{aligned}$$

Just as in the original proof by Dawid, Hartmann and Sprenger, combining this set of equations with the fact that $\sum_{k=0}^{\infty} y_k = \sum_{k=0}^{\infty} P\{Y = k\} = 1$, we arrive at

$$\begin{aligned}
\tilde{\Delta} &= \sum_{i=0}^{\infty} t_i \sum_{j=0}^{\infty} g_{ij} y_i \sum_{k=0}^{\infty} y_k - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} y_i g_{ij} \sum_{k=0}^{\infty} t_k y_k \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} y_i y_k g_{ij} \left(t_i - t_k\right). \qquad (1)
\end{aligned}$$

Now, it is quite obvious that for all $I, J, K \in \mathbf{N}$,

$$\begin{aligned}
& \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} |y_i y_k g_{ij} (t_i - t_k)| = \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} y_i y_k g_{ij} \underbrace{|t_i - t_k|}_{\leq 1} \\
\leq\ & \sum_{i=0}^{I} \sum_{j=0}^{J} y_i \underbrace{\left(\sum_{k=0}^{K} y_k \right)}_{\leq 1} g_{ij} \leq \sum_{i=0}^{I} \sum_{j=0}^{J} y_i g_{ij} \\
=\ & \sum_{i=0}^{I} \sum_{j=0}^{J} P\left\{Y = i\right\} P\left(F_A \cap \{D = j\} \mid \{Y = i\}\right) \\
=\ & \sum_{i=0}^{I} \sum_{j=0}^{J} P\left(F_A \cap \{D = j\} \cap \{Y = i\}\right) \\
=\ & P\left(\bigcup_{i=0}^{I} \bigcup_{j=0}^{J} \left(F_A \cap \{D = j\} \cap \{Y = i\}\right) \right) \\
\leq\ & P\left(F_A\right) \leq 1,
\end{aligned}$$

whence the partial sums of the iterated series in Equation (1) are all uniformly bounded. By well-known results on iterated infinite series (cf. e.g. [3, p. 69f.: Hauptkriterium für Summierbarkeit, Großer Umordnungssatz]), the series may be reordered.

The rest of the proof is almost identical to the original derivation by Dawid, Hartmann and Sprenger [2, Appendix B, p. 16f.], but for the sake of completeness, it is included here. First, it is obvious that

$$\tilde{\Delta} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\substack{k=0 \\ k \neq i}}^{\infty} y_i y_k g_{ij} \left(t_i - t_k\right), \qquad (2)$$

and the main step consists then in reordering

$$S_j := \sum_{i=0}^{\infty} \sum_{\substack{k=0 \\ k \neq i}}^{\infty} y_i y_k g_{ij} \left(t_i - t_k\right).$$

The first reordering yields

$$S_j = \sum_{i=0}^{\infty} \sum_{k>i} y_i y_k g_{ij} \left(t_i - t_k\right) + y_i y_k g_{kj} \left(t_k - t_i\right). \tag{3}$$

Then, one may again rearrange, in order to obtain $S_j = \sum_{k=0}^{\infty} \sum_{i<k} y_i y_k g_{ij} \left(t_i - t_k\right) + y_i y_k g_{kj} \left(t_k - t_i\right)$ and (after changing variables)

$$S_j = \sum_{i=0}^{\infty} \sum_{k<i} y_i y_k g_{kj} \left(t_k - t_i\right) + y_i y_k g_{ij} \left(t_i - t_k\right).$$

Adding the last equation to Equation (3), one arrives at

$$2S_j = \sum_{i=0}^{\infty} \sum_{k \neq i} y_i y_k g_{ij} \left(t_i - t_k\right) + y_i y_k g_{kj} \left(t_k - t_i\right).$$

Resubstituting S_j, we conclude that for all $j \in \mathbf{N}$,

$$\sum_{i=0}^{\infty} \sum_{\substack{k=0 \\ k \neq i}}^{\infty} y_i y_k g_{ij} \left(t_i - t_k\right) \quad = \quad \frac{1}{2} \sum_{i=0}^{\infty} \sum_{\substack{k=0 \\ k \neq i}}^{\infty} y_i y_k g_{ij} \left(t_i - t_k\right) + y_i y_k g_{kj} \left(t_k - t_i\right)$$

and thus by Equation (2),

$$\begin{aligned}
\tilde{\Delta} &= \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\substack{k=0 \\ k \neq i}}^{\infty} y_i y_k g_{ij} \left(t_i - t_k\right) + y_i y_k g_{kj} \left(t_k - t_i\right) \\
&= \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\substack{k=0 \\ k \neq i}}^{\infty} y_i y_k \left(g_{ij} \left(t_i - t_k\right) + g_{kj} \left(t_k - t_i\right)\right) \\
&= \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\substack{k=0 \\ k \neq i}}^{\infty} y_i y_k \left(t_i - t_k\right) \left(g_{ij} - g_{kj}\right) \\
&= \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\substack{k=0 \\ k \neq i}}^{\infty} y_i y_k \left(t_i - t_k\right) \left(\tilde{d}_j \tilde{f}_{ij} - \tilde{d}_j \tilde{f}_{kj}\right) \\
&= \frac{1}{2} \sum_{j=0}^{\infty} \tilde{d}_j \sum_{i=0}^{\infty} \sum_{\substack{k=0 \\ k \neq i}}^{\infty} y_i y_k \left(t_i - t_k\right) \left(\tilde{f}_{ij} - \tilde{f}_{kj}\right),
\end{aligned}$$

which is strictly positive in light of assumptions A3', A4 and A5'.

On the other hand,

$$\tilde{\Delta} = (P(T|F_A) - P(T))\, P(F_A),$$

whence it follows that $P(T|F_A) - P(T) > 0$.

□

Proof of Remark 2. In the special case of Y being independent of D, one may simply choose $\tilde{d}_j = P\{D = j\}$ (which by independence equals $P(\{D = j\}|\{Y = k\})$ for all k) and $\tilde{f}_{kj} = P(F_A|\{Y = k\} \cap \{D = j\})$. Then, clearly

$$\begin{aligned} \tilde{d}_j \tilde{f}_{kj} &= P(\{D = j\}|\{Y = k\})\, P(F_A|\{Y = k\} \cap \{D = j\}) \\ &= P(F_A\,|\{Y = k\}\,, \{D = j\}\,) \end{aligned}$$

and the monotonicity requirements of A3' and A5' follow directly from those in A3 and A5. □

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