First-Order Extensions of Classical Modal Logic

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Horacio's Neighborhood

Horacio Arló-Costa (2002). *First Order Extensions of Classical Systems of Modal Logic: The role of the Barcan Schemas*. Studia Logica, 71:1, pgs. 87 - 118.

Horacio Arló-Costa (2005). *Non-Adjunctive Inference and Classical Modalities*. Journal of Philosophical Logic, 34:5, pgs. 581 - 605.

Horacio Arló-Costa and EP (2006). *First-Order Classical Modal Logic*. Horacio Arlo-Costa and Eric Pacuit, Studia Logica, Volume 84:2, pgs. 171 - 210.

Plan

1. Background
   - Neighborhood Semantics for Propositional Modal Logic
   - First-Order Modal Logic
   - The Barcan Formula

2. Neighborhood Models for First-Order Modal Logic

H. Arló-Costa and E. Pacuit. *First-Order Classical Modal Logic*. Studia Logica, **84**, pgs. 171 - 210 (2006).

4. General Frames for First-Order Modal Logic

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4. General Frames for First-Order Modal Logic

R. Goldblatt and E. Mares. *A General Semantics for Quantified Modal Logic*. AiML, 2006.

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

What does it mean to be a neighborhood?

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

neighborhood in some topology.

J. McKinsey and A. Tarski. *The Algebra of Topology*. 1944.

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contains all the immediate neighbors in some graph

S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

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S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

an element of some distinguished collection of sets

D. Scott. *Advice on Modal Logic*. 1970.

R. Montague. *Pragmatics*. 1968.

Classical Modal Logic

To see the necessity of the more general approach, we could consider probability operators, conditional necessity, or, to invoke an especially perspicuous example of Dana Scott, the present progressive tense....Thus N might receive the awkward reading 'it is being the case that', in the sense in which 'it is being the case that Jones leaves' is synonymous with 'Jones is leaving'. (Montague, 1970)

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$

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A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

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E is the smallest classical modal logic.

In **E**, *M* is equivalent to

(*Mon*) $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

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EM is the logic **E** + *Mon*

PC 6. Propositional Calculus

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A logic is normal if it contains all instances of *E*, *C* and is closed under *Mon* and *Nec*

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K $= PC(+E) + K + Nec + MP$

R. Goldblatt. *Mathematical Modal Logic: A View of its Evolution*. Handbook of the History of Logic, 2005.

Logics of High Probability

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H. Kyburg and C.M. Teng. *The Logic of Risky Knowledge*. Proceedings of WoLLIC (2002).

A. Herzig. *Modal Probability, Belief, and Actions*. Fundamenta Informaticae (2003).

Neighborhood Frames

Let W be a non-empty set of states.

Any map $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N\rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model is a tuple $\langle W, N, V\rangle$ where $V : \mathrm{At} \to \wp(W)$ is a valuation function and $\langle W, N\rangle$ is a neighborhood frame.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

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- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\varphi$ iff $(\varphi)^{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - (\varphi)^{\mathfrak{M}} \not\in N(w)$

where $(\varphi)^{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Validities

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ is valid in all neighborhood models.

(Re) If $\varphi \leftrightarrow \psi$ is valid then $\Box\varphi \leftrightarrow \Box\psi$ is valid.

Horacio's Possibility

$\mathfrak{M}, w \models \Box\varphi$ iff $(\varphi)^{\mathfrak{M}} \in N(w)$

$\mathfrak{M}, w \models \Diamond\varphi$ iff $W - (\varphi)^{\mathfrak{M}} \notin N(w)$

$\mathfrak{M}, w \models \Box\varphi$ iff there is an $X \in N(w)$ such that for all $v \in W$, $v \in X$ iff $\mathfrak{M}, v \models \varphi$

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- $\mathfrak{M}, w \models \langle\,\rangle\varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\,]\varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$

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Other Examples, I

Reasoning about abilities

M. Brown. *On the Logic of Ability*. Journal of Philosophical Logic, 17, p. 1 - 26 (1988).

Reasoning about games

R. Parikh. *The Logic of Games and its Applications*. Annals of Discrete Mathematics (1985).

Reasoning about coalitions

M. Pauly. *Logic for Social Software*. Ph.D. Thesis, ILLC (2001).

Other Examples, II

Epistemic Logic: the logical omniscience problem.

M. Vardi. *On Epistemic Logic and Logical Omniscience*. TARK (1986).

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M. Vardi. *On Epistemic Logic and Logical Omniscience*. TARK (1986).

Reasoning about evidence and beliefs (and how they change over time)

J. van Benthem and EP. *Dynamics of Evidence-Based Beliefs*. Studia Logica, 2011.

J. van Benthem, D. Fernández-Duque, EP. *Evidence Logic: A New Look and Neighborhood Structures*. Advances in Modal Logic, 2012.

Other Examples, III

Program logics: modeling concurrent programs

D. Peleg. *Concurrent Dynamic Logic*. J. ACM (1987).

The Logic of deduction

P. Naumov. *On modal logic of deductive closure*. APAL (2005).

Deontic logics...

Constraints on neighborhood frames

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- **Contains the unit:** $W \in N(w)$

Constraints on neighborhood frames

- **Monotonic or Supplemented:** If $X \in N(w)$ and $X \subseteq Y$, then $Y \in N(w)$
- **Closed under finite intersections:** If $X \in N(w)$ and $Y \in N(w)$, then $X \cap Y \in N(w)$
- **Contains the unit:** $W \in N(w)$
- **Augmented:** Supplemented plus for each $w \in W$, $\bigcap N(w) \in N(w)$

Coherent Neighborhoods

A neighborhood of w is "perfectly coherent" provided $\bigcap N(w) \neq \emptyset$,

Coherent Neighborhoods

A neighborhood of w is "perfectly coherent" provided $\bigcap N(w) \neq \emptyset$, how should we measure the "level of coherence" when $N(w)$ is not closed under conjunction?

Horacio Arló-Costa (2005). *Non-Adjunctive Inference and Classical Modalities*. Journal of Philosophical Logic, 34:5, pgs. 581 - 605.

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

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Definition

Given a relation R on a set W and a state $w \in W$. A set $X \subseteq W$ is R-necessary at w if $R^{\rightarrow}(w) \subseteq X$.

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Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

$$\mathcal{N}_w^R = \{X \mid R^{\rightarrow}(w) \subseteq X\}$$

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Lemma

Let R be a relation on W. Then for each $w \in W$, $\mathcal{N}_w^R$ is augmented.

From Neighborhood Frames to Kripke Frames

Theorem

- *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*
- *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

From Neighborhood Frames to Kripke Frames

for all $X \subseteq W$, $X \in N(w)$ iff $X \in \mathcal{N}_w^R$.

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- *Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

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Proof.

For each $w \in W$, let $N(w) = \mathcal{N}^R_w$. □

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Theorem

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✓ *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.

For each $w, v \in W$, $wR_N v$ iff $v \in \cap N(w)$. □

Definability Results

1. $\mathcal{F} \models \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ iff $\mathcal{F}$ is closed under supersets (monotonic frames).

2. $\mathcal{F} \models \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \Box\psi)$ iff $\mathcal{F}$ is closed under finite intersections.

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4. $\mathcal{F} \models$ **EMCN** iff $\mathcal{F}$ is a filter

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3. $\mathcal{F} \models \Box\top$ iff $\mathcal{F}$ contains the unit
4. $\mathcal{F} \models$ **EMCN** iff $\mathcal{F}$ is a filter
5. $\mathcal{F} \models \Box\varphi \rightarrow \varphi$ iff for each $w \in W$, $w \in \cap N(w)$
6. And so on...

Completeness Results

- **E** is sound and strongly complete with respect to the class of **all** neighborhood frames
- **EM** is sound and strongly complete with respect to the class of all **monotonic** neighborhood frames
- **EC** is sound and strongly complete with respect to the class of all neighborhood frames that are **closed under finite intersections**
- **EN** is sound and strongly complete with respect to the class of all neighborhood frames that **contain the unit**
- **K** is sound and strongly complete with respect to the class of all neighborhood frames that are **filters**
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Some Results

- For each Kripke model $\langle W, R, V\rangle$, there is an pointwise equivalent *augmented* neighborhood model $\langle W, N, V\rangle$.
- Bimodal normal modal logics can simulate non-normal modal logics (Kracht and Wolter 1999)
- There are logics which are complete with respect to a class of neighborhood frames but not complete with respect to relational frames (D. Gabbay 1975, M. Gerson 1975, M. Gerson 1976).
- There are logics incomplete with respect to neighborhood frames (M. Gerson, 1975; T. Litak, 2005).
- Model theory of classical modal logics: Bisimulation, van Benthem Characterization Theorem, Interpolation, etc. (H. Hansen, C. Kupke and EP, 2010)

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- There are logics which are complete with respect to a class of neighborhood frames but not complete with respect to relational frames (D. Gabbay 1975, M. Gerson 1975, M. Gerson 1976).
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First-order extensions

First-Order Modal Language: $\mathcal{L}_1$

Extend the propositional modal language $\mathcal{L}$ with the usual first-order machinery (constants, terms, predicate symbols, quantifiers).

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$$A := P(t_1, \ldots, t_n) \mid \neg A \mid A \wedge A \mid \Box A \mid \forall x A$$

(note that equality is not in the language!)

State-of-the-art

T. Braüner and S. Ghilardi. *First-order Modal Logic*. Handbook of Modal Logic, pgs. 549 - 620 (2007).

D.Gabbay, V. Shehtman and D. Skvortsov. *Quantification in Nonclassical Logic*. Elsevier, 2009.

http://lpcs.math.msu.su/~shehtman/n.ps

M. Fitting and R. Mendelsohn. *First-Order Modal Logic*. Kluwer Academic Publishers (1998).

First-order Modal Logic

A **constant domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where W and D are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $V(P, w) \subseteq D^n$.

A **substitution** is any function $\sigma : \mathcal{V} \rightarrow D$ ($\mathcal{V}$ the set of variables).

A substitution σ' is said to be an x-**variant** of σ if $\sigma(y) = \sigma'(y)$ for all variable y except possibly x, this will be denoted by $\sigma \sim_x \sigma'$.

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Suppose that σ is a substitution.

1. $\mathcal{M}, w \models_\sigma P(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n)\rangle \in V(P, w)$
2. $\mathcal{M}, w \models_\sigma \Box A$ iff $R(w) \subseteq (A)^{\mathcal{M},\sigma}$
3. $\mathcal{M}, w \models_\sigma \forall x A$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} A$

First-order Modal Logic

A **constant domain Neighborhood frame** is a tuple $\langle W, N, D\rangle$ where W and D are sets, and $N : W \to \wp(\wp(W))$.

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2. $\mathcal{M}, w \models_\sigma \Box A$ iff $(A)^{\mathcal{M},\sigma} \in N(w)$
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First-order Modal Logic

Let **S** be any (classical) propositional modal logic, by $\mathbf{FOL} + \mathbf{S}$ we mean the set of formulas closed under the following rules and axioms:

(S) All instances of axioms and rules from **S**.

($\forall$) $\forall x A \rightarrow A^x_t$ (where t is free for x in A)

(Gen) $\dfrac{A \rightarrow B}{A \rightarrow \forall x B}$, where x is not free in A.

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$
- **converse Barcan formula** (CBF): $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

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- **converse Barcan formula** (*CBF*): $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

Observation 1: *CBF* is provable in **FOL** + **EM**

Observation 2: *BF* and *CBF* both valid on relational frames with constant domains

Observation 3: *BF* is valid in a *varying* domain relational frame iff the frame is anti-monotonic; *CBF* is valid in a *varying* domain relational frame iff the frame is monotonic.

See (Fitting and Mendelsohn, 1998) for an extended discussion

Constant Domains without the Barcan Formula

The system **EMN** and seems to play a central role in characterizing monadic operators of high probability (See Kyburg and Teng 2002, Arló-Costa 2004).

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Of course, *BF* should fail in this case, given that it instantiates cases of what is usually known as the '**lottery paradox**':

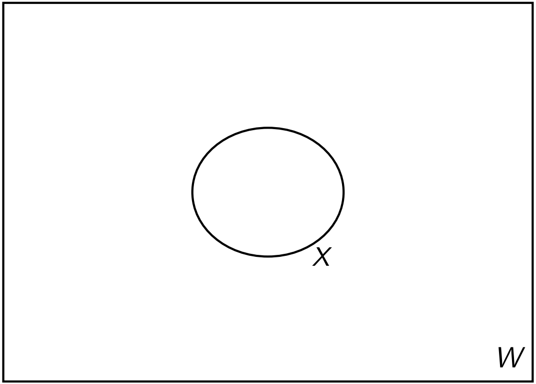
For each individual x, it is *highly probably* that x will loose the lottery; however it is not necessarily highly probably that each individual will loose the lottery.

Converse Barcan Formulas and Neighborhood Frames

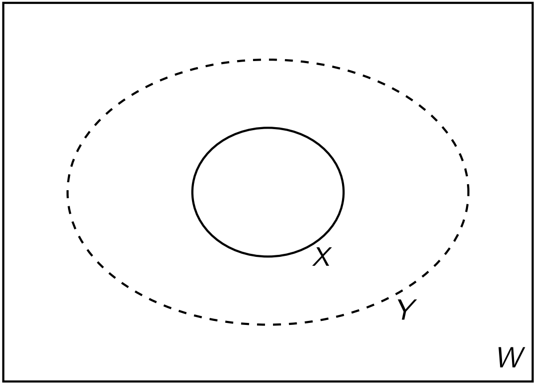
A frame $\mathcal{F}$ is **consistent** iff for each $w \in W$, $N(w) \neq \emptyset$

A first-order neighborhood frame $\mathcal{F} = \langle W, N, D \rangle$ is **nontrivial** iff $|D| > 1$

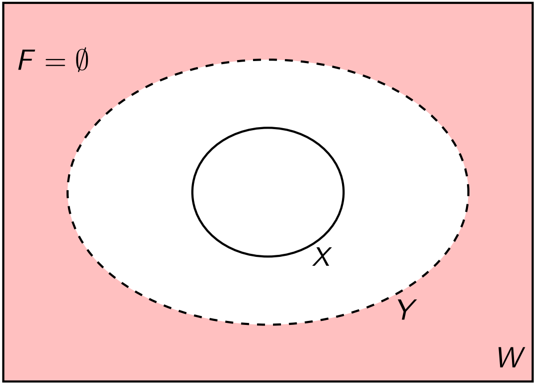
Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The converse Barcan formula is valid on $\mathcal{F}$ iff either $\mathcal{F}$ is trivial or $\mathcal{F}$ is supplemented.



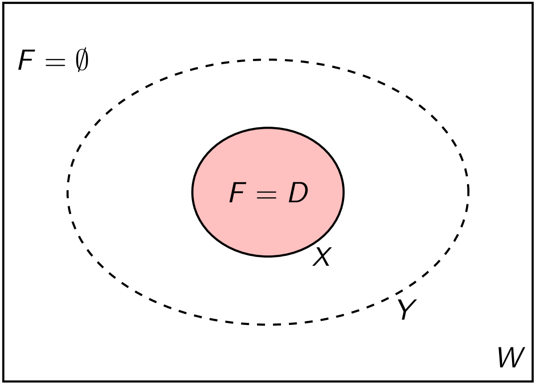
$$X \in N(w)$$



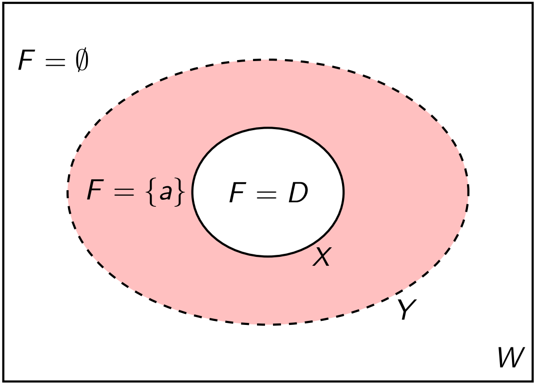
$$Y \not\in N(w)$$



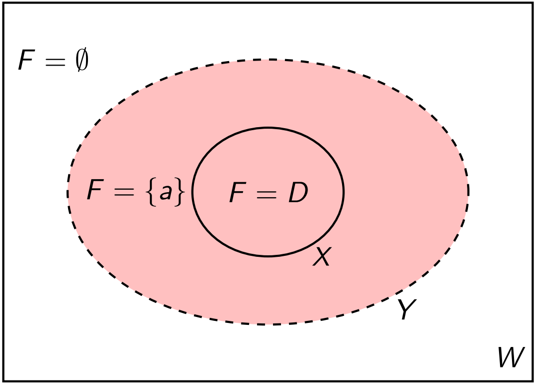
$$\forall v \notin Y, \quad I(F, v) = \emptyset$$



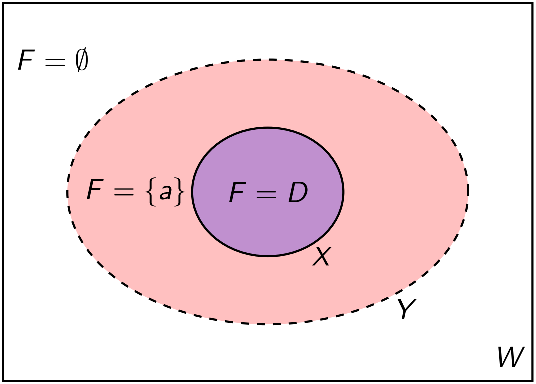
$$\forall v \in X, \quad I(F, v) = D = \{a, b\}$$



$$\forall v \in Y - X, \quad I(F, v) = D = \{a\}$$



$$(F[a])^{\mathcal{M}} = Y \not\in N(w) \quad \text{hence} \quad w \not\models \forall x \Box F(x)$$



$$(\forall x F(x))^{\mathcal{M}} = (F[a])^{\mathcal{M}} \cap (F[b])^{\mathcal{M}} = X \in N(w)$$

$$\text{hence} \quad w \models \Box \forall x F(x)$$

Barcan Formulas and Neighborhood Frames

We say that a frame closed under $\leq \kappa$ intersections if for each state w and each collection of sets $\{X_i \mid i \in I\}$ where $|I| \leq \kappa$, $\cap_{i\in I} X_i \in N(w)$.

Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The Barcan formula is valid on $\mathcal{F}$ iff either

1. $\mathcal{F}$ is trivial or
2. if D is finite, then $\mathcal{F}$ is closed under finite intersections and if D is infinite and of cardinality κ, then $\mathcal{F}$ is closed under $\leq \kappa$ intersections.

Completeness Theorems

Theorem $\mathbf{FOL} + \mathbf{E}$ is sound and strongly complete with respect to the class of **all** frames.

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Theorem **FOL** + **EM** is sound and strongly complete with respect to the class of supplemented frames.

Theorem **FOL** + **E** + CBF is sound and strongly complete with respect to the class of frames that are either non-trivial and supplemented or trivial and not supplemented.

FOL + **K** and **FOL** + **K** + *BF*

Theorem **FOL** + **K** is sound and strongly complete with respect to the class of filters.

FOL + K and FOL + K + *BF*

Theorem **FOL** + **K** is sound and strongly complete with respect to the class of filters.

Observation The augmentation of the smallest canonical model for **FOL** + **K** is not a canonical model for **FOL** + **K**. In fact, the closure under infinite intersection of the minimal canonical model for **FOL** + **K** is not a canonical model for **FOL** + **K**.

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Lemma The augmentation of the smallest canonical model for **FOL** + **K** + *BF* is a canonical for **FOL** + **K** + *BF*.

Theorem **FOL** + **K** + *BF* is sound and strongly complete with respect to the class of augmented first-order neighborhood frames.

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1. **S4M** is complete for the class of all frames that are reflexive, transitive and *final* (every world can see an 'end-point'). However **FOL** + **S4M** is incomplete for Kripke models based on **S4M**-frames. (see Hughes and Cresswell, pg. 283).

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2. **S4.2** is S4 with $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$. This logics is complete for the class of frames that are reflexive, transitive and *convergent*. However, **FOL** + **S4M** + *BF* is incomplete for the class of constant domain models based on reflexive, transitive and convergent frames. (see Hughes and Cresswell, pg. 271)

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3. The quantified extension of **GL** is not recursively axiomatizable (Cresswell, 1997).

What is going on?

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Horacio Arló-Costa and EP (2006). *First-Order Classical Modal Logic*. Horacio Arlo-Costa and Eric Pacuit, Studia Logica, Volume 84:2, pgs. 171 - 210.

R. Goldblatt and E. Mares. *A General Semantics for Quantified Modal Logic*. AiML, 2006.

R. Goldblatt. *Quantifiers, Propositions and Identity: Admissible Semantics for Quantified Modal and Substructural Logics*. Lecture Notes in Logic No. 38, Cambridge University Press and the Association for Symbolic Logic, 2011.

▸ Skip

Background: Incompleteness

There are (consistent) modal logics that are incomplete

A general model is a structure $\langle W, R, V, \mathcal{A}\rangle$ where $\mathcal{A}$ is a suitable boolean algebra with an operator of propositions.

All modal logics are sound and strongly complete with respect to general frames.

Theorem (Goldblatt and Mares) For any **canonical** propositional modal logic **S**, its quantified extension **QS** is complete over a class of general frames for which the underlying propositional frame are just the **S**-frames.

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- New perspective on the Barcan formula: it corresponds to **Tarskian models**
- There is a trade-off between having the underlying Kripke frame validate the propositional logic in question and having a Tarskian-reading of the quantifier.

Central Idea

Algebraic reading of the universal quantifier: $\forall x \varphi$ is true at a world w iff there is some proposition X such that X entails every instantiation of φ and X obtains at w.

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$\mathcal{M}, w \models_\sigma \forall x A$ iff there is a proposition X such that $w \in X$ and $X \subseteq (A)^{\mathcal{M}}_{\sigma(x|d)}$ for all $d \in D$.

vs.

$\mathcal{M}, w \models_\sigma \forall x A$ iff for all $d \in D$, $\mathcal{M}, w \models_{\sigma(x|d)} A$

General Frames

Let $\langle W, R\rangle$ be a frame.

$[R] : \wp W \to \wp W$ where
$[R](X) = \{w \in W \mid \text{ for all } v \in W,\ wRv \text{ implies } v \in X\}$
So $(\Box\alpha)^{\mathcal{M}} = [R](\alpha)^{\mathcal{M}}$

$X \Rightarrow Y = (W - X) \cup Y$
So $(\alpha \to \beta)^{\mathcal{M}} = (\alpha)^{\mathcal{M}} \Rightarrow (\beta)^{\mathcal{M}}$.

Halmos Functions

$\varphi : D^{\mathcal{V}} \to \wp W$
Let φ and ψ be two such functions, we can lift $[R]$ and $\Rightarrow$ to operations of functions: Eg., if $\varphi : D^{\mathcal{V}} \to \wp W$ and $f \in D^{\mathcal{V}}$.
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Fix a set $Prop \subseteq \wp W$. This defines for each $S \subseteq \wp W$,

$$\sqcap S = \bigcup \{X \in Prop \mid X \subseteq \bigcap S\}$$

General Frames for First-Order Modal Logic

Suppose $Prop \subseteq \wp W$ and let $\varphi : D^{\mathcal{V}} \to Prop$,
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$\langle W, R, V, Prop, PropFun \rangle$ where

- *Prop* contains $\emptyset$ and is closed under $\Rightarrow$ and $[R]$
- Contains the function $\varphi_\emptyset(f) = \emptyset$ for all $f \in D^{\mathcal{V}}$
- PropFun is closed under $\Rightarrow$, $[R]$ and $\forall_x$.
- Assume $(P)^{\mathcal{M}} : D^{\mathcal{V}} \to \wp W$ is an element of *PropFun* for each atomic predicate P.

General Completeness

Theorem For any propositional modal logic **S**, the quantified logic **QS** is complete for the class of (all validating) quantified general frames.

Note that the canonical model construction has as worlds maximally consistent sets that need not be $\forall$*-complete*.

Key Results

Theorem (Goldblatt and Mares) If **S** is a canonical propositional logic, then **QS** is characterized by the class of all **QS**-frames whose underlying propositional frames validate **S**.

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Logics containing the Barcan formula have **two** characterizing canonical general frames: one that is Tarskian and one that is not.

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Logics containing the Barcan formula have **two** characterizing canonical general frames: one that is Tarskian and one that is not.

1. If **S** is canonical, then the second canonical model will have an underlying propositional frame that validates **S** (eg., **S4.2**), but may not be Tarskian.
2. On the other hand, The Tarskian canonical model may not have an underlying propositional frame that is a frame for **S** (again **S4.2** is an example).

R. Goldblatt. *Quantifiers, Propositions and Identity: Admissible Semantics for Quantified Modal and Substructural Logics*. Lecture Notes in Logic No. 38, Cambridge University Press and the Association for Symbolic Logic, 2011.

Conclusions

- Characterize other "modality-quantifier interactions" in terms of properties on neighborhoods (eg. $\exists x \Box \varphi(x) \rightarrow \Box \exists x \varphi(x)$)

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G. Boella, D. Gabbay, V. Genovese, L. van der Torre. *Higher-Order Coalition Logic*. Proceedings of ECAI, 2010.

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G. Boella, D. Gabbay, V. Genovese, L. van der Torre. *Higher-Order Coalition Logic*. Proceedings of ECAI, 2010.

- Different perspectives on the first-order modal language.

H. Sturm and F. Wolter. *First-order Expressivity for S5-models: Modal vs. two-sorted Languages*. Journal of Philosophical Logic (2000).

Thank you.

Extensions: Higher-Order Coalition Logic

G. Boella, D. Gabbay, V. Genovese, L. van der Torre. *Higher-Order Coalition Logic*. 2010.

Strategy Logics

- *Coalitional Logic*: Reasoning about (local) group power.

  $[C]\varphi$: coalition C has a **joint action** to bring about φ.

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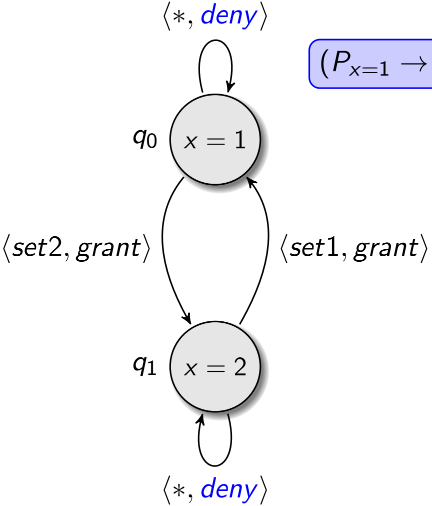
- *Alternating-time Temporal Logic*: Reasoning about (local and global) group power:

  $\langle\langle A\rangle\rangle\Box\varphi$: The coalition A has a **joint action** to ensure that φ will remain true.

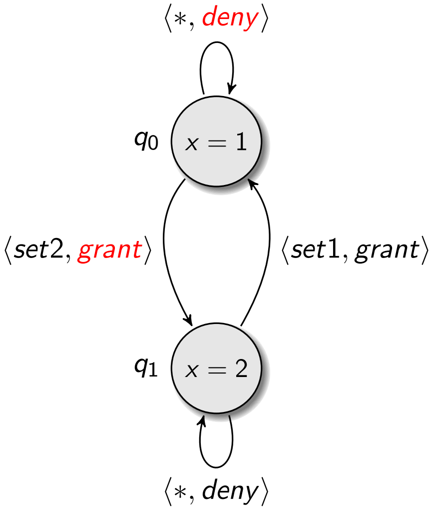
R. Alur, T. Henzinger and O. Kupferman. *Alternating-time Temproal Logic*. *Jouranl of the ACM* (2002).

Multi-agent Transition Systems



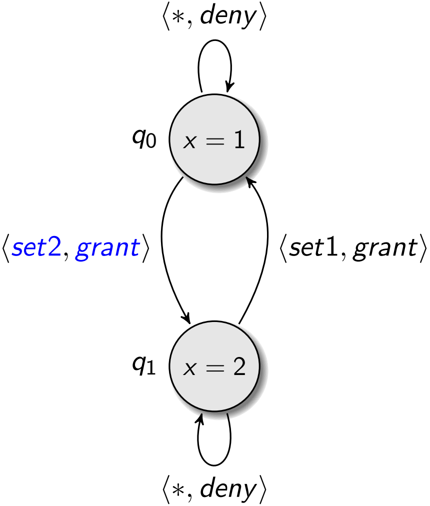
$(P_{x=1} \rightarrow [s]P_{x=1}) \wedge (P_{x=2} \rightarrow [s]P_{x=2})$

Multi-agent Transition Systems



$P_{x=1} \rightarrow \neg[s]P_{x=2}$

Multi-agent Transition Systems



$P_{x=1} \to [s, c]P_{x=2}$

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Higher-Order Coalition Logic: $\varphi :=$
$F(x_1, \ldots, x_n) \mid Xx \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall X\varphi \mid \forall x\varphi \mid [\{x\}\varphi]\varphi \mid \langle\{x\}\varphi\rangle\varphi$

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- $F(x_1, \ldots, x_n)$ is a first-order atomic formula
- x is a first-order variable
- X is a set variable
- $\{x\}\psi$ is a group operator representing the set of all d such that $\psi[d/x]$ holds

HCL: Expressivity

What does the added expressive power give you?

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- Relationships between coalitions:
  $\forall x(super\_user(x) \rightarrow user(x))$

- General quantification over coalitions:
  $\forall X(\forall x(Xx \rightarrow user(x)) \rightarrow [\{y\}Xy]\varphi)$

  Every coalition such that all of its members are users can achieve φ*.*

HCL: Expressivity

What does the added expressive power give you?

- Relationships between coalitions:
  $\forall x(super\_user(x) \rightarrow user(x))$
- General quantification over coalitions:
  $\forall X(\forall x(Xx \rightarrow user(x)) \rightarrow [\{y\}Xy]\varphi)$

  Every coalition such that all of its members are users can achieve φ.
- Complex relationships between coalitions and agents:
  $[\{x\}\varphi(x)]\psi \rightarrow [\{y\}\exists x(\varphi(x) \wedge collaborates(y,x))]\psi$

  If the coalition represented by φ can achieve ψ then so can any group that collaborates with at least one member of $\varphi(x)$.

HCL: Barcan/Converse Barcan Formulas

Converse Barcan: $[\{x\}\varphi(x)]\forall y\psi(y) \rightarrow \forall y[\{x\}\varphi(x)]\varphi(y)$

Barcan: $\forall y[\{x\}\varphi(x)]\varphi(y) \rightarrow [\{x\}\varphi(x)]\forall y\psi(y)$

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$\forall y[\{x\}x = Eric](CMU(y) \rightarrow happy(y)) \not\rightarrow$
$[\{x\}x = Eric]\forall y(CMU(y) \rightarrow happy(y))$

For each person at CMU, I can make them happy does not imply that I can do something to make everyone at CMU happy.

Higher-Order Coalition Logic

Sound and complete axiomatization combines ideas from coalition logic, first-order extensions of non-normal modal logics and Henkin-style completeness for second-order logic.

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