- Naïvely, "the problem of logical omniscience" (for Bayesian epistemology) is generated *via* the following argument:
- Bayesians assume various things about epistemically rational agents, including the following (and more later):
  - Epistemically rational agents have quantitative degrees of belief (credences) in statements [and credences play an important role in epistemology – more on their role later].
  - Credences of epistemically rational agents can be modeled using *probability functions* [*epistemic states* of rational agents can be represented using *probability models*].
- Probability theory implies that "probability functions assign equal probabilities to logically equivalent statements."
- ∴ Bayesian epistemology implies that epistemically rational agents must assign equal credences to logically equivalent statements. So, *Bayesianism implies logical omniscience*.
- Does this mean that Bayesians can't model logically *non*-omniscient agents? No. Today, I'll try to explain why.

- We will need some technical background about the probability calculus. But, I'll try to keep it to a minimum.
- A *probability model* $\mathcal{M}$ consists of a finite (*i.e.*, finitely many atomic sentences) sentential language P, together with a function $\Pr : P \mapsto \mathbb{R}$ such that for all sentences p and q in P:
  - $\Pr(p) \geq 0$.
  - If $p \dashv\vDash_P \top$, then $\Pr(p) = 1$.
  - If $p \,\&\, q \dashv\vDash_P \bot$, then $\Pr(p \vee q) = \Pr(p) + \Pr(q)$.
  - $\Pr(p \mid q) \overset{\text{def}}{=} \frac{\Pr(p\&q)}{\Pr(q)}$, provided that $\Pr(q) \neq 0$.
- We will see some salient examples of $\mathcal{M}$'s in a few slides.
- **Fact.** If $p \dashv\vDash_P q$, then $\Pr(p) = \Pr(q)$. *This* is the precise theoretical sense in which "all probability functions assign equal probability to logically equivalent statements."
- **Definition.** *E confirms H relative to (or in) a probability model $\mathcal{M}$* (where $E, H \in P$) just in case $\Pr(H \mid E) > \Pr(H)$.
- **Fact.** If $\Pr(E) = 1$ (in $\mathcal{M}$), then E can't confirm *any* H (in $\mathcal{M}$).
- This fact gives rise to "The Problem of Old Evidence" [7].

- Naïvely, the problem of old evidence is generated *via* three "orthodox" Bayesian *epistemic modeling assumptions*:
  (1) The epistemic state of a rational agent a at a time t can be faithfully characterized by **a** probability model $\mathcal{M}_t^a$.
  (2) All confirmational judgments a makes at t must *supervene* on $\mathcal{M}_t^a$. More precisely, a may judge (at t) that E confirms H *only if* E confirms H *relative to* $\mathcal{M}_t^a$.
  (3) If a knows that E at t, then (in $\mathcal{M}_t^a$) $\Pr(E) = 1$.

  Assumptions (1)–(3) + **Fact** lead us to an odd conclusion:
  (4) If a knows that E at t, then (at t) a may *not* judge that E confirms H (and this holds for *any* H in $\mathcal{M}_t^a$).
- Most Bayesians (myself included) respond by denying (2) [2].
- Some recommend expanding the supervenience base in (2) to include *historical* ($t' < t$) epistemic Pr-models $\mathcal{M}_{t'}^a$.
- Others advise expanding (2)'s SB to include *counterfactual* (*e.g.*, a' is a *counterpart* of a) epistemic Pr-models $\mathcal{M}_t^{a'}$.
- I think we need to expand (2)'s SB to include *objective* probability models, but *that's* a story for another talk [4]!

- The canonical example of "the problem of old evidence" involves Einstein, GTR, and the perihelion of Mercury.
- Einstein (a) knew in 1915 (t) – and this had *long* been known [13] – that (E) the perihelion of Mercury advances at $\approx 43''$ of arc per century (above and beyond the precession already predicted by Newtonian theory). Thus, in $\mathcal{M}_t^a$, $\Pr(E) = 1$.
- But, contrary to (4), Einstein does (in 1915) *seem* to judge that E confirms H (GTR+Auxiliaries), and this *seems* to be a reasonable judgment for Einstein to have made at that time.
- As I said, most Bayesians try to find a way to reject (2) here. I have my own way to reject (2) *via* IL [4]. I won't get into it.
- Garber [6] and Jeffrey [9] *accept* that Einstein should *not* have judged (in 1915) that E confirms H. They offer a different explanation of Einstein's confirmational judgment.
- Idea: Einstein *did* know E at t, but he *didn't* know (at t) that "*H entails E*" (he was not "logically omniscient"). So, while E *couldn't* have confirmed H (at t, for a), "*H entails E*" *could*.

- Next: Garber's "logical learning" approach to "old evidence". But, first: "logical ignorance" and Bayesian coherence.
- There are (at least) *three grades* of logical ignorance:
  - (LI_1) Ignorance of some logical relations *in* P caused by a having a false conception of the nature of logic itself.
    - In our present context, this would involve $\dashv\vDash_P$ being an incorrect theoretical explication of logical equivalence in P.
  - (LI_2) Ignorance of some logical relations *external to* P, reflected in "representational impoverishment" of P.
    - P is given an *extrasystematic interpretation* (involving some richer theory T), which obscures some *extrasystematic* entailments ($\vDash_T$'s). [No *systematic* $\vDash_P$-ignorance here!]
  - (LI_3) Ignorance of some logical relations *in* P caused by error, laziness, computational/intellectual limitations, *etc.*
    - This involves a failing (at t) to recognize some classical tautological equivalences *in* P (*i.e.*, *systematic* $\vDash_P$-ignorance).
- ☞ Of these three grades, ***only*** (LI_3) *can be a cause of classical Bayesian* ***incoherence*** (vulnerability to "Dutch Book" [10])!
- I will focus on (LI_2). Few Bayesians worry about (LI_1) [14].

- Garber rejects the "global reading" of (1). He argues that *various* "local" probability models may be appropriate for modeling *various aspects of* the epistemic state of a at t.
- Garber proposes a class of probability models for the purpose of modeling *certain aspects of* Einstein's epistemic state in 1915 [including his (LI_2)-ignorance of $H \vDash_T E$].
- Garber's models $\mathcal{G}$ involve a language P with four atomic statements: A, B, C, D. Initially, A–D are uninterpreted and so *any* credence function Pr over P is rationally permissible.
- Next, Garber *extrasystematically interprets* A as H (GTRA), B as E (mercury data), C as $H \vDash_T E$, and D as $H \vDash_T {\sim}E$. The basic conjunctions of P then become *candidate epistemic possibilities for* a *at* t, and Pr encodes a's *credences* at t.
- "p is epistemically possibile for a at t" $\stackrel{\text{def}}{=}$ "it is permissible for a to assign $\Pr(p) > 0$ at t." In this sense, *both* $H \vDash_T E$ *and* $H \nvDash_T E$ were "possibilities" for Einstein in 1915.
- Now we're ready to see what Garber's models $\mathcal{G}$ look like ...

A	B	C	D	Pr		H	E	$H \vDash_T E$	$H \vDash_T {\sim}E$	$\Pr_{1915}^{\text{Einstein}}$
$\top$	$\top$	$\top$	$\top$	$p_1 \in [0,1]$		$\top$	$\top$	$\top$	$\top$	0
$\top$	$\top$	$\top$	$\bot$	$p_2 \in [0,1]$		$\top$	$\top$	$\top$	$\bot$	$p \in (0,1)$
$\top$	$\top$	$\bot$	$\top$	$p_3 \in [0,1]$		$\top$	$\top$	$\bot$	$\top$	0
$\top$	$\top$	$\bot$	$\bot$	$p_4 \in [0,1]$		$\top$	$\top$	$\bot$	$\bot$	$q \in (0,1)$
$\top$	$\bot$	$\top$	$\top$	$p_5 \in [0,1]$		$\top$	$\bot$	$\top$	$\top$	0
$\top$	$\bot$	$\top$	$\bot$	$p_6 \in [0,1]$		$\top$	$\bot$	$\top$	$\bot$	0
$\top$	$\bot$	$\bot$	$\top$	$p_7 \in [0,1]$		$\top$	$\bot$	$\bot$	$\top$	0
$\top$	$\bot$	$\bot$	$\bot$	$p_8 \in [0,1]$	$\mapsto$	$\top$	$\bot$	$\bot$	$\bot$	0
$\bot$	$\top$	$\top$	$\top$	$p_9 \in [0,1]$		$\bot$	$\top$	$\top$	$\top$	$r \in (0,1)$
$\bot$	$\top$	$\top$	$\bot$	$p_{10} \in [0,1]$		$\bot$	$\top$	$\top$	$\bot$	$s \in (0,1)$
$\bot$	$\top$	$\bot$	$\top$	$p_{11} \in [0,1]$		$\bot$	$\top$	$\bot$	$\top$	$t \in (0,1)$
$\bot$	$\top$	$\bot$	$\bot$	$p_{12} \in [0,1]$		$\bot$	$\top$	$\bot$	$\bot$	$u \in (0,1)$
$\bot$	$\bot$	$\top$	$\top$	$p_{13} \in [0,1]$		$\bot$	$\bot$	$\top$	$\top$	0
$\bot$	$\bot$	$\top$	$\bot$	$p_{14} \in [0,1]$		$\bot$	$\bot$	$\top$	$\bot$	0
$\bot$	$\bot$	$\bot$	$\top$	$p_{15} \in [0,1]$		$\bot$	$\bot$	$\bot$	$\top$	0
$\bot$	$\bot$	$\bot$	$\bot$	$p_{16} \in [0,1]$		$\bot$	$\bot$	$\bot$	$\bot$	0

- Garber's aim was simply to describe models $\mathcal{G}$ in which $H \vDash_T E$ *could* confirm H, *even though* $\Pr(E) = 1$ in $\mathcal{G}$.
- Garber does not provide us with specific *constraints* on $p, \ldots, u$ which would *entail* that $\Pr(H \mid H \vDash_T E) > \Pr(H)$.
- Jeffrey [9] and Earman [1] pick-up where Garber leaves off, and they each give sufficient conditions in this sense:
  - **Jeffrey**: $\Pr(H \vDash_T E \vee H \vDash_T {\sim}E) = 1$, *i.e.*, $q = 0$ and $u = 0$. Then, $\Pr(H \mid H \vDash_T E) > \Pr(H)$ reduces to $\frac{p}{p+r+s} > p$, which follows from $p, \ldots, u \in (0,1)$ and $p + q + r + s + t + u = 1$.
  - **Earman**: $\Pr(H \mid H \vDash_T E) > \Pr(H \mid H \nvDash_T E \,\&\, H \nvDash_T {\sim}E)$. Algebraically, this reduces to: $\frac{p}{p+r+s} > \frac{q}{q+u}$. Non-trivially, this entails $\frac{p}{p+r+s} > p + q$ [*viz.*, $\Pr(H \mid H \vDash_T E) > \Pr(H)$], because $p, \ldots, u \in (0,1)$ and $p + q + r + s + t + u = 1$.
- Earman's constraint is plausible (for Einstein in 1915).
- This gives Garber a plausible story: In 1915, Einstein learned that $H \vDash_T E$ (this is also plausible [13]), and it was *this* which boosted Einstein's credence in H. E did not provide any boost (in 1915), since he already knew it.

- As any classical Bayesian must, Garber is assuming that Einstein is omniscient in sense (LI_3). That is, he is assuming omniscience about $\dashv\vDash_P$, where P is the language of $\mathcal{G}$.
- Garber also assumes Einstein has *a modicum of* (high-level) knowledge about $\vDash_T$. This (incomplete!) extrasystematic logical knowledge is reflected in $\mathcal{G}$'s probability function Pr.
- Garber uses an idealized, "local" probability model over P to model learning logical relations in T. Modeling P-logical learning would (presumably) require *another* "local" model.

- Is Garber's "extrasystematic interpretation" of P (inducing "extrasystematic relations" between P's *atoms* to partially reflect logical relations *external to P*) *kosher*? Well, it had better be!

☞ Historically, this is a *central* Bayesian technique ([3],[11]).

- Paradox? Q: How can $H \vDash_T E$ and $H \nvDash_T E$ both be *epistemic* possibilities for a when a *knows* they *can't* both be *logical* possibilities? A: Not all "epistemic possibilities" (in our Garberian sense) express logically possible propositions!

- Good [8] suggests an alternative "evolving probability" (EP) approach, which requires only that *known* logical truths get credence 1, and *known* incompatibles satisfy additivity.
- (EP) is *ambiguous* between grades (LI_2) and (LI_3) ignorance. On an (LI_3) reading, EP amounts to weakening $p \dashv\vDash_P q$ to "a *knows* that $p \dashv\vDash_P q$ at t" [*i.e.*, $K(p \dashv\vDash_P q)$] in the Pr-axioms.
- [Is $K(p \dashv\vDash_P q)$ is an *equivalence relation*? If not, then $\mathcal{M}$ isn't a probability model. See [5] for a nice discussion, and a rigorous, proof-theoretic alternative to Garber and (EP).]
- On *either* reading, EP recommends that we change (*only*!) our credences so as to "reflect the learned logical relation".
  - *E.g.*, if we learn "H entails E", then EP prescribes adopting a new credence function $\Pr'$ such that $\Pr'(E \mid H) = 1$.
- Garber critiques EP. His main complaint: EP can't handle OE *via* logical learning, since in OE cases we *already* have $\Pr(E \mid H) = 1$ *before* a learns $H \vDash_T E$. So, in OE cases, EP can't account for *any* Pr shift arising from learning $H \vDash_T E$.
- I think Garber is right. But, I want to say a bit more here.

- Kukla [12] defends EP against Garber. But, he concedes:

  > On the EP account, ... prior to [learning the logical fact in question], our probability function was incoherent in the classical Bayesian sense. Indeed, our probability functions are *always* incoherent ... and no doubt always will be.

- This is a brave concession! And, a mistake. ***Incoherence*** requires (LI_3)-ignorance. Making this grade of ignorance so ubiquitous just *trivializes* the notion of "coherence".
- *Pace* Good *and* Garber (& Jeffrey), I suspect we can't always repair (capture) all effects (aspects) of logical ignorance (learning) at t *merely* by changing our Pr over our old P.
- Some examples of logical learning seem to involve moving to *new language* P', which can articulate logical relations obscured in P. This goes beyond previous approaches.
- Garber models varying degrees of logical ignorance about T by tweaking his extrasystematic interpretation of P. This is clever, and an improvement over EP. But, I think this still obscures a salient kind/facet of change in epistemic state.

- OK, so the question is: if we sometimes want not only a Pr-shift but also a P-shift (*i.e.*, a *model* shift), then what principles should guide us in formulating our new model?
- This is not an easy question. But, I have a few ideas.
- Like Garber, I suggest adding atoms to the (naïve) model (language) so as to capture obscured logical structure.
- But, I suggest: (a) do this *diachronically*, and (b) reflect the learned relations as *tautological relations in P' itself* (*vs* "$\vDash_T$-relations" in an *extrasystematic interpretation of P*).
- *E.g.*, when a learns $H \vDash_T E$, we might model a as moving to a new $\mathcal{M}'$ [$\langle P', \Pr' \rangle$] in which what was expressed in P as "H" (GTRA) is now expressed as "$E \,\&\, X$", for a new X in P'.
- If this is an OE case, then $\Pr(E)$ and $\Pr'(E)$ will both equal 1. Thus, $\Pr'(E \,\&\, X) = \Pr'(X)$. So, the "probability boost" Garber wants reduces to $\Pr'(X) > \Pr(H)$. *I.e.*, in old-evidence cases:

  ☞ Learning $H \vDash_T E$ between t and t' boosts a's credence in H if the part of H that "goes beyond" E (represented by a new "X" in P') has greater credence for a at t' than H had at t.

- Garber introduces a new proposition $[C]$ into his algebra, which "expresses" $H \vDash_T E$. This allows Garber to model the (logical) learning of $H \vDash_T E$ *via conditionalization on* C.
- Jeffrey rejects conditionalization (as a rule) for various reasons. (*i*) We shouldn't (always) assign probability 1 to learned contingents. (*ii*) There isn't always a proposition in one's algebra which expresses that which one has learned.
- In OE cases of *logical learning* (*i*) is moot (here, all *is* learned *with certainty*). (*ii*) doesn't apply to the (empirical) learning of E. Perhaps (*ii*) applies to the (logical) learning of $H \vDash_T E$.
- But, why *couldn't* there be a statement in P "expressing" $H \vDash_T E$? No wff in P *systematically* expresses $H \vDash_T E$. So?
- *That **can't*** be *required* of a conditionalization approach ([3], [11]). In any case, Jeffrey abandons conditionalization here.
- One more issue: Like all the others, Jeffrey models logical learning events as updates of Pr's over *fixed languages*.
- Putting these worries aside, here's Jeffrey's approach …

- Jeffrey [9, Postscript] operates with a more parsimonious language $P_{\mathcal{J}}$, containing just two atomic statements: H, E.
- He models the learning of $H \vDash_T E$ (logical) and E (emprical) as (Jeffrey!) updates on Pr's over $P_{\mathcal{J}}$: $\Pr^0 \mapsto \Pr^1 \mapsto \Pr^2$.
- We begin with what Jeffrey calls the "ur-function" $\Pr^0$, which assigns non-extreme credence to each basic conj. of $P_{\mathcal{J}}$.
- Provisionally, Jeffrey has a do their logical update "first". He assumes two things about this $\Pr^0 \mapsto \Pr^1$ logical update:
  (9) $\Pr^1(H) = \Pr^0(H)$. *Why*? "Learning that H implies something that may well be false neither confirms nor infirms H".
  (10) $\Pr^1(H \,\&\, {\sim}E) = 0$. *Why*? Because a learned $H \vDash_T E$ here!
- Jeffrey-updating subject to (9)&(10) yields a *unique* $\Pr^1$ from $\Pr^0$:

H	E	$\Pr^0$
⊤	⊤	$a \in (0,1)$
⊤	⊥	$b \in (0,1)$
⊥	⊤	$c \in (0,1)$
⊥	⊥	$d \in (0,1)$

$H \vDash_T E$ $\longmapsto$

H	E	$\Pr^1$
⊤	⊤	$a + b \in (0,1)$
⊤	⊥	0
⊥	⊤	$c \in (0,1)$
⊥	⊥	$d \in (0,1)$

- "Next", the empirical update (E) occurs. Jeffrey models this as learning E *with certainty*. [Yep, by ***conditionalizing*** $\Pr^1$ on E! He must do so *otherwise the OEP does not arise*. Not very "radical"!]
- In all, Jeffrey makes 3 more assumptions about the 2 updates:
  (11) $\Pr^2(\cdot) = \Pr^1(\cdot \mid E)$. *Why*? E is learned *with certainty* in OE.
  (12) $\Pr^0(H \mid E) = \Pr^0(H)$. *Why*? Unclear (simplifies the math).
  (13) The logical and empirical updates should *commute* — the order in which they come should not have an effect on $\Pr^2$. *Why*? Otherwise, E cannot be "old" when $H \vDash_T E$ is learned.
- Jeffrey-updating subject to (9)–(13) yields a *unique* $\Pr^2$ from $\Pr^0$:

H	E	$\Pr^2$
⊤	⊤	$\frac{a+b}{a+b+c} \in (0,1)$
⊤	⊥	0
⊥	⊤	$\frac{c}{a+b+c} \in (0,1)$
⊥	⊥	0

- Since $\frac{a+b}{a+b+c} > a + b$, $\Pr^2(H) > \Pr^0(H)$, and H has received a "probability boost" from learning (E *and then*) $H \vDash_T E$.

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