Philosophy 148 — Assignment #1

02/14/08

This assignment is due Thursday, 2/28/08. Answer all questions. If you work in a group, list your group members at the top of your submitted work.

1 Problem #1

In this class, we take a probability function $\Pr(\bullet)$ to be a real-valued function on a (essentially, sentential) language $\mathcal{L}$ containing finitely many atomic sentences satisfying the following three (*Kolmogorov*) axioms, for all sentences $p, q \in \mathcal{L}$:

1. $\Pr(p) \geq 0$.
2. If $p \Dashv\vDash \top$, then $\Pr(p) = 1$.
3. If $p \,\&\, q \Dashv\vDash \bot$, then $\Pr(p \vee q) = \Pr(p) + \Pr(q)$.

In Ch. 6 of his book *Choice and Chance: An Introduction to Inductive Logic* (posted on website), Skyrms adopts the following six probability "rules" :

(*i*) If $p \Dashv\vDash \top$, then $\Pr(p) = 1$.

(*ii*) If $p \Dashv\vDash \bot$, then $\Pr(p) = 0$.

(*iii*) If $p \Dashv\vDash q$, then $\Pr(p) = \Pr(q)$.

(*iv*) If $p \,\&\, q \Dashv\vDash \bot$, then $\Pr(p \vee q) = \Pr(p) + \Pr(q)$.

(*v*) $\Pr({\sim}p) = 1 - \Pr(p)$.

(*vi*) $\Pr(p \vee q) = \Pr(p) + \Pr(q) - \Pr(p \,\&\, q)$.

Here are the two problems you must solve:

(*a*) Prove Skyrms's six rules from our axioms. Specifically, prove all of his six rules from our axioms (2) and (3) alone. You may not use any results proved in class until you've proved them (yourself) as "lemmas" for the purpose of this problem. You may prove Skyrms's rules in any order, and once you have proved a result you may use it in subsequent proofs.

(*b*) Which of our axioms follow from Skyrms's rules and which do not? Explain this in as much detail as you can (but in your own words).

(*c*) Give an example of a claim that (i) follows from our axioms, (ii) is *not* one of our axioms, and (iii) does *not* follow from Skyrms's six rules. Prove that the example you give does in fact follow from our axioms but not from Skyrms's six rules

The theorems you have proved in this problem may be used in subsequent problems and on subsequent homeworks without retracing their proofs.

2 Problem #2

Here is a deductively valid argument:

> **Argument 1:** If X is true, then Z is true. If Y is true, then Z is true. Therefore, if $(X \vee Y)$ is true, Z is true.

Here is an argument about conditional probabilities that *looks* analogous:

> **Argument 2:** Conditional on X, the probability of Z is q. Conditional on Y, the probability of Z is q. Therefore, conditional on $(X \vee Y)$, the probability of Z is q. More formally, the argument is:

$$\begin{aligned} &\Pr(Z \mid X) = q. \\ &\Pr(Z \mid Y) = q. \\ \therefore\ &\Pr(Z \mid X \vee Y) = q. \end{aligned}$$

(*a*) Give a complete probability model that provides a counter-example to Argument 2. A complete probability model specifies a probability for each possible state-description. (If you work in a group, each member of the group must give a different probability model.)

(*b*) Show that your model is in fact a counter-example to Argument 2.

Extra Credit [worth 5%]. Give a "real-world" example of a probabilistic situation (sampling, *etc.*) that provides a counter-example to Argument 2. The probabilities in your example need not be the same as in your model from part *a*.

3 Problem #3

I mentioned in class that human beings tend to look for very simple kinds of probability models. These usually involve non-regular probability distributions, and also ones that involve *rational*-valued basic probabilities. In this problem, I will present a very simple set of probabilistic statements that do not have any "simple" models, in either sense. Here is the set of conditions I have in mind:

1. $\Pr(Y \mid X) = \Pr(Y \vee X)$.
2. $\Pr(X \,\&\, Y) = \dfrac{1}{4}$.
3. $\Pr({\sim}X \,\&\, Y) = \dfrac{1}{4}$.

Fact. There is only one probability model $\mathcal{M}$ on which (1)–(3) are all true. And, $\mathcal{M}$ is a *regular* probability model containing some *irrational* basic probabilities.

Your goal in this problem is to *prove* that **Fact** is true. **Hint**. It helps to represent the class of possible models using the following stochastic truth-table:

X	Y	$\Pr(s_i)$
T	T	$\frac{1}{4}$
T	F	x
F	T	$\frac{1}{4}$
F	F	$1 - (\frac{1}{4} + x + \frac{1}{4}) = \frac{1}{2} - x$

Now, all you need to do is express condition (1) in terms of the single unknown x, and then explain why condition (1) determines a unique probability model $\mathcal{M}$ (*i.e.*, a unique value for x), such that $\mathcal{M}$ satisfies the description in **Fact**.

4 Problem #4

In class, we claimed that if three propositions are mutually independent on a given model, then one is independent from any Boolean combination of the other two. Prove the following case of that theorem, either from the Kolmogorov axioms (you may use theorems you have proved in previous problems), or using algebraic methods (translating the theorem into algebraic terms using a stochastic truth-table representation of a 3-atomic-proposition space).

Theorem. If $\{X, Y, Z\}$ are mutually independent, then X and $Y \equiv Z$ are independent. That is, *if* the following four conditions obtain:

- $\Pr(X \,\&\, Y) = \Pr(X) \cdot \Pr(Y)$
- $\Pr(X \,\&\, Z) = \Pr(X) \cdot \Pr(Z)$
- $\Pr(Y \,\&\, Z) = \Pr(Y) \cdot \Pr(Z)$
- $\Pr(X \,\&\, Y \,\&\, Z) = \Pr(X) \cdot \Pr(Y) \cdot \Pr(Z)$

then, the following condition also obtains:

- $\Pr(X \,\&\, (Y \equiv Z)) = \Pr(X) \cdot \Pr(Y \equiv Z)$

[For what it's worth, I think this one may be easier to do *axiomatically*!]