

PrSAT: First Examples

Philosophy 148

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■ First, load in the `PrSAT` package

See my [PrSAT website](#) for instructions on downloading and installing `PrSAT` (assuming you have *Mathematica* installed).

```
In[1]:= << PrSAT`
```

■ Example #1

The first example of a probability model that we saw was the following:

```
In[2]:= MODEL1 = PrSAT[{{Pr[X & Y] == 1/6, Pr[X & ~Y] == 1/4, Pr[~X & Y] == 1/8, Pr[~X & ~Y] == 11/24}}]
```

```
Out[2]:= {{X -> {a2, a4}, Y -> {a3, a4}, Omega -> {a1, a2, a3, a4}}, {a1 -> 11/24, a2 -> 1/4, a3 -> 1/8, a4 -> 1/6}}
```

`PrSAT` will show us an STT representation of `MODEL1`:

```
In[3]:= TruthTable[MODEL1]
```

Out[3]/DisplayForm=

X	Y	var	Pr
T	T	a ₄	$\frac{1}{6}$
T	F	a ₂	$\frac{1}{4}$
F	T	a ₃	$\frac{1}{8}$
F	F	a ₁	$\frac{11}{24}$

We can use `PrSAT` to calculate probability, using `MODEL1`:

```
In[4]:= EvaluateProbability[{Pr[X ∨ Y], Pr[X], Pr[Y]}, MODEL1]
```

```
Out[4]:= {13/24, 5/12, 7/24}
```

We can also check arbitrary claims to see if they are *true on MODEL1*:

```
In[5]:= EvaluateProbability[Pr[X | Y] > Pr[X], MODEL1]
```

```
Out[5]:= True
```

■ Example #2

The second example we saw was an algebraic proof of the following theorem:

```
In[6]:= Pr(X ∨ Y) = Pr(X) + Pr(Y) - Pr(X ∧ Y)
```

PrSAT easily verifies this theorem (note: it does not present a readable proof).

```
In[7]:= PrSAT[ {Pr[X ∨ Y] ≠ Pr[X] + Pr[Y] - Pr[X ∧ Y] } ]
```

```
PrSAT::srchfail : Search phase failed; attempting FindInstance
```

```
Out[7]= { }
```

This output means there are no probability models on which $\Pr[X \vee Y] \neq \Pr[X] + \Pr[Y] - \Pr[X \wedge Y]$. That “proves” that the above statement is a theorem of probability calculus.

■ Example #3

The second example we saw was an algebraic proof of the following theorem:

```
In[8]:= Pr(X) = Pr(X ∧ Y) + Pr(X ∧ ¬ Y)
```

PrSAT easily verifies this theorem (note: it does not present a readable proof).

```
In[8]:= PrSAT[ {Pr[X] ≠ Pr[X ∧ Y] + Pr[X ∧ ¬ Y] } ]
```

```
PrSAT::srchfail : Search phase failed; attempting FindInstance
```

```
Out[8]= { }
```

This output means there are no probability models on which $\Pr[X] \neq \Pr[X \wedge Y] + \Pr[X \wedge \neg Y]$. That “proves” that the above statement is a theorem of probability calculus.

■ Example #4

The next example involves the following theorem:

```
In[10]:= Pr(X → Y) ≥ Pr(Y | X)
```

PrSAT easily verifies this theorem (note: it does not present a readable proof). First, we need to define the conditional operator.

```
In[9]:= X_ → Y_ := ¬ X ∨ Y;
```

```
In[10]:= PrSAT[ {Pr[X → Y] < Pr[Y | X] } ]
```

```
PrSAT::srchfail : Search phase failed; attempting FindInstance
```

```
Out[10]= { }
```

This output means there are no probability models on which $\Pr[X \rightarrow Y] \geq \Pr[Y | X]$. That “proves” that the above statement is a theorem of probability calculus.

■ Example #5

The next example involves the following theorem:

In[13]:= $d(X, Y) = d(X \vee Y, Y) + d(X \vee \neg Y, Y)$, where $d(X, Y) = \Pr(X|Y) - \Pr(X)$.

PrSAT easily verifies this theorem (note: it does not present a readable proof). First, we need to define $\mathbf{d}(\mathbf{x}, \mathbf{y})$.

In[11]:= $\mathbf{d}[\mathbf{x}_-, \mathbf{y}_-] := \Pr[\mathbf{x} | \mathbf{y}] - \Pr[\mathbf{x}]$;

In[12]:= **PrSAT**[{ $\mathbf{d}[\mathbf{x}, \mathbf{y}] \neq \mathbf{d}[\mathbf{x} \vee \mathbf{y}, \mathbf{y}] + \mathbf{d}[\mathbf{x} \vee \neg \mathbf{y}, \mathbf{y}]$ }]

PrSAT::srchfail : Search phase failed; attempting FindInstance

Out[12]= {}

This output means there are no probability models on which $\mathbf{d}[\mathbf{x}, \mathbf{y}] \neq \mathbf{d}[\mathbf{x} \vee \mathbf{y}, \mathbf{y}] + \mathbf{d}[\mathbf{x} \vee \neg \mathbf{y}, \mathbf{y}]$. That “proves” that the above statement is a theorem of probability calculus.

■ Example #6

The next example involves the fact that the following is NOT a theorem:

In[14]:= $\Pr(X|Y \vee Z) = \Pr(X|Y \wedge Z)$

PrSAT easily finds a counter-model to this claim.

In[13]:= **PrSAT**[{ $\Pr[\mathbf{x} | \mathbf{y} \vee \mathbf{z}] \neq \Pr[\mathbf{x} | \mathbf{y} \wedge \mathbf{z}]$ }]

Out[13]= $\left\{ \left\{ \begin{array}{l} X \rightarrow \{a_2, a_5, a_6, a_8\}, Y \rightarrow \{a_3, a_5, a_7, a_8\}, \\ Z \rightarrow \{a_4, a_6, a_7, a_8\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}, \\ \left\{ a_1 \rightarrow 0, a_2 \rightarrow 0, a_3 \rightarrow 0, a_4 \rightarrow 0, a_5 \rightarrow \frac{1}{28}, a_6 \rightarrow \frac{13}{28}, a_7 \rightarrow \frac{1}{2}, a_8 \rightarrow 0 \right\} \right\} \right\}$

The model **PrSAT** finds by default is *non-regular*. We can force it to find a *regular* counter-model, as follows:

In[14]:= **MODEL2 = PrSAT**[{ $\Pr[\mathbf{x} | \mathbf{y} \vee \mathbf{z}] \neq \Pr[\mathbf{x} | \mathbf{y} \wedge \mathbf{z}]$ }, **Probabilities** \rightarrow **Regular**]

Out[14]= $\left\{ \left\{ \begin{array}{l} X \rightarrow \{a_2, a_5, a_6, a_8\}, Y \rightarrow \{a_3, a_5, a_7, a_8\}, \\ Z \rightarrow \{a_4, a_6, a_7, a_8\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}, \\ \left\{ a_1 \rightarrow \frac{147181}{77577345}, a_2 \rightarrow \frac{1}{999}, a_3 \rightarrow \frac{46}{93}, a_4 \rightarrow \frac{1}{405}, a_5 \rightarrow \frac{1}{999}, a_6 \rightarrow \frac{1}{999}, a_7 \rightarrow \frac{1}{999}, a_8 \rightarrow \frac{83}{167} \right\} \right\} \right\}$

Here is an STT representation of **MODEL2**:

In[15]:= **TruthTable[MODEL2]**

Out[15]//DisplayForm=

X	Y	Z	var	Pr
T	T	T	a ₈	$\frac{83}{167}$
T	T	F	a ₅	$\frac{1}{999}$
T	F	T	a ₆	$\frac{1}{999}$
T	F	F	a ₂	$\frac{1}{999}$
F	T	T	a ₇	$\frac{1}{999}$
F	T	F	a ₃	$\frac{46}{93}$
F	F	T	a ₄	$\frac{1}{405}$
F	F	F	a ₁	$\frac{147181}{77577345}$

We can calculate the values of $\Pr[X \mid Y \wedge Z]$, $\Pr[X \mid Y \vee Z]$ on this model as follows:

In[16]:= **EvaluateProbability**[{**Pr**[X | Y ∧ Z], **Pr**[X | Y ∨ Z]}, **MODEL2**]

Out[16]= $\left\{ \frac{82917}{83084}, \frac{38711715}{77352509} \right\}$

We can look at decimal representations of these exact real numbers, as follows:

In[17]:= **% // N**

Out[17]= {0.99799, 0.500458}

We gave a different model in the lecture notes. We can enter that model in by hand, and then verify it has the desired properties, as follows:

In[18]:= **MODEL3 = PrSAT**[{**Pr**[X ∧ Y ∧ Z] == $\frac{1}{6}$, **Pr**[X ∧ Y ∧ ¬Z] == $\frac{1}{6}$, **Pr**[X ∧ ¬Y ∧ Z] == $\frac{1}{4}$, **Pr**[X ∧ ¬Y ∧ ¬Z] == $\frac{1}{16}$,
Pr[¬X ∧ Y ∧ Z] == $\frac{1}{6}$, **Pr**[¬X ∧ Y ∧ ¬Z] == $\frac{1}{12}$, **Pr**[¬X ∧ ¬Y ∧ Z] == $\frac{1}{24}$, **Pr**[¬X ∧ ¬Y ∧ ¬Z] == $\frac{1}{16}$ }]

Out[18]= $\left\{ \{X \rightarrow \{a_2, a_5, a_6, a_8\}, Y \rightarrow \{a_3, a_5, a_7, a_8\},\right.$
 $Z \rightarrow \{a_4, a_6, a_7, a_8\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\},$
 $\left. \left\{ a_1 \rightarrow \frac{1}{16}, a_2 \rightarrow \frac{1}{16}, a_3 \rightarrow \frac{1}{12}, a_4 \rightarrow \frac{1}{24}, a_5 \rightarrow \frac{1}{6}, a_6 \rightarrow \frac{1}{4}, a_7 \rightarrow \frac{1}{6}, a_8 \rightarrow \frac{1}{6} \right\} \right\}$

In[19]:= **TruthTable[MODEL3]**

Out[19]//DisplayForm=

X	Y	Z	var	Pr
T	T	T	a_8	$\frac{1}{6}$
T	T	F	a_5	$\frac{1}{6}$
T	F	T	a_6	$\frac{1}{4}$
T	F	F	a_2	$\frac{1}{16}$
F	T	T	a_7	$\frac{1}{6}$
F	T	F	a_3	$\frac{1}{12}$
F	F	T	a_4	$\frac{1}{24}$
F	F	F	a_1	$\frac{1}{16}$

In[20]:= **EvaluateProbability**[{**Pr**[X | Y ∧ Z], **Pr**[X | Y ∨ Z]}, **MODEL3**]

Out[20]= $\left\{ \frac{1}{2}, \frac{2}{3} \right\}$

We can also see the algebraic form of an expression, as follows:

In[21]:= **AlgebraicForm**[**Pr**[X | Y ∧ Z] == **Pr**[X | Y ∨ Z], {X, Y, Z}]

Out[21]=
$$\frac{a_8}{a_7 + a_8} == \frac{a_5 + a_6 + a_8}{a_3 + a_4 + a_5 + a_6 + a_7 + a_8}$$

Note that **PrSAT** uses different conventions (*i.e.*, a different ordering in the truth-table) for the a_i than I use in the lecture notes.