PrSAT: Second Examples

Philosophy 148

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First, load in the `PrSAT` package

See my **`PrSAT` website** for instructions on downloading and installing **`PrSAT`** (assuming you have *Mathematica* installed).

```
<< PrSAT`
```

Example #0

Proving a couple of equivalence theorems about independence. First, define a non-equivalence operator (for simpler input):

```
In[45]:= X_ ≢ Y_ := LogicalExpand[Not[(X ∧ Y) || (! X ∧ ! Y)]];

In[51]:= PrSAT[{
           (Pr[X ∧ Y] == Pr[X] Pr[Y]) ≢ (Pr[X ∧ ¬ Y] == Pr[X] Pr[¬ Y])
          }]

         PrSAT::srchfail : Search phase failed; attempting FindInstance

Out[51]= { }

In[52]:= PrSAT[
          {
           (Pr[X ∧ Y] == Pr[X] Pr[Y]) ≢ (Pr[¬ X ∧ ¬ Y] == Pr[¬ X] Pr[¬ Y])
          }]

         PrSAT::srchfail : Search phase failed; attempting FindInstance

Out[52]= { }
```

Example #1

The pairwise independence *vs* mutual independence counterexample involving the labeled tickets.

```
MODEL1 = PrSAT[
  {
   Pr[X] == 1/2,
   Pr[Y] == 1/2,
   Pr[Z] == 1/2,
   Pr[X ∧ Y ∧ Z] == 0,
   Pr[X ∧ Y] == 1/4,
   Pr[X ∧ Z] == 1/4,
   Pr[Y ∧ Z] == 1/4
  }]
```

$\{\{X \to \{a_2, a_5, a_6, a_8\}, Y \to \{a_3, a_5, a_7, a_8\}, Z \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \{a_1 \to \frac{1}{4}, a_2 \to 0, a_3 \to 0, a_4 \to 0, a_5 \to \frac{1}{4}, a_6 \to \frac{1}{4}, a_7 \to \frac{1}{4}, a_8 \to 0\}\}$

```
TruthTable[MODEL1]
```

X	Y	Z	var	Pr
T	T	T	a_8	0
T	T	F	a_5	$\frac{1}{4}$
T	F	T	a_6	$\frac{1}{4}$
T	F	F	a_2	0
F	T	T	a_7	$\frac{1}{4}$
F	T	F	a_3	0
F	F	T	a_4	0
F	F	F	a_1	$\frac{1}{4}$

We can use **EvaluateProbability** to check that this model is a counterexample to the claim in question:

```
EvaluateProbability[{
  Pr[X ∧ Y] == Pr[X] Pr[Y],
  Pr[X ∧ Z] == Pr[X] Pr[Z],
  Pr[Y ∧ Z] == Pr[Y] Pr[Z],
  Pr[X ∧ Y ∧ Z] == Pr[X] Pr[Y] Pr[Z]},
 MODEL1]
```

{True, True, True, False}

We can use **PrSAT** to find a *regular* countermodel to this claim, as follows:

```
MODEL2 = PrSAT[{
   Pr[X ∧ Y] == Pr[X] Pr[Y],
   Pr[X ∧ Z] == Pr[X] Pr[Z],
   Pr[Y ∧ Z] == Pr[Y] Pr[Z],
   Pr[X ∧ Y ∧ Z] ≠ Pr[X] Pr[Y] Pr[Z]}, Probabilities → Regular]
```

$$\left\{\{X \to \{a_2, a_5, a_6, a_8\}, Y \to \{a_3, a_5, a_7, a_8\}, Z \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \left\{a_1 \to \frac{84418 - 39\sqrt{4676097}}{56277}, a_2 \to \frac{-42296 + 39\sqrt{4676097}}{168831}, a_3 \to \frac{-42296 + 39\sqrt{4676097}}{168831}, a_4 \to \frac{-42296 + 39\sqrt{4676097}}{168831}, a_5 \to \frac{1}{999}, a_6 \to \frac{1}{999}, a_7 \to \frac{1}{999}, a_8 \to \frac{42}{169}\right\}\right\}$$

```
TruthTable[MODEL2]
```

X	Y	Z	var	Pr
T	T	T	a_8	$\frac{42}{169}$
T	T	F	a_5	$\frac{1}{999}$
T	F	T	a_6	$\frac{1}{999}$
T	F	F	a_2	$\frac{-42296+39\sqrt{4676097}}{168831}$
F	T	T	a_7	$\frac{1}{999}$
F	T	F	a_3	$\frac{-42296+39\sqrt{4676097}}{168831}$
F	F	T	a_4	$\frac{-42296+39\sqrt{4676097}}{168831}$
F	F	F	a_1	$\frac{84418-39\sqrt{4676097}}{56277}$

■ Example #2

A counterexample to transitivity of independence. We can use **PrSAT** to automatically find such a counterexample:

```
MODEL3 = PrSAT[
  {
   Pr[X ∧ Y] == Pr[X] Pr[Y],
   Pr[Y ∧ Z] == Pr[Y] Pr[Z],
   Pr[X ∧ Z] ≠ Pr[X] Pr[Z]
  }, Probabilities → Regular]
```

$$\left\{\{X \to \{a_2, a_5, a_6, a_8\}, Y \to \{a_3, a_5, a_7, a_8\}, Z \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \left\{a_1 \to \frac{1}{999}, a_2 \to \frac{427}{285714}, a_3 \to \frac{71}{143}, a_4 \to \frac{427}{285714}, a_5 \to \frac{1}{999}, a_6 \to \frac{1}{999}, a_7 \to \frac{1}{999}, a_8 \to \frac{71}{143}\right\}\right\}$$

```
TruthTable[MODEL3]
```

X	Y	Z	var	Pr
T	T	T	a_8	$\frac{71}{143}$
T	T	F	a_5	$\frac{1}{999}$
T	F	T	a_6	$\frac{1}{999}$
T	F	F	a_2	$\frac{427}{285714}$
F	T	T	a_7	$\frac{1}{999}$
F	T	F	a_3	$\frac{71}{143}$
F	F	T	a_4	$\frac{427}{285714}$
F	F	F	a_1	$\frac{1}{999}$

We can use **EvaluateProbability** to check that this model is a counterexample to the claim in question:

```
EvaluateProbability[{
  Pr[X ∧ Y] == Pr[X] Pr[Y],
  Pr[Y ∧ Z] == Pr[Y] Pr[Z],
  Pr[X ∧ Z] == Pr[X] Pr[Z]},
 MODEL3]
```

{True, True, False}

■ Example #3

An example of *Simpson's Paradox*:

```
MODEL4 = PrSAT[
  {
   Pr[p ⋀ q | r] == Pr[p | r] Pr[q | r],
   Pr[p ⋀ q | ¬ r] == Pr[p | ¬ r] Pr[q | ¬ r],
   Pr[p ⋀ q] ≠ Pr[p] Pr[q]
  }, Probabilities → Regular]
```

$\{\{p \to \{a_2, a_5, a_6, a_8\}, q \to \{a_3, a_5, a_7, a_8\},$
$r \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\},$
$\{a_1 \to \frac{12}{91}, a_2 \to \frac{5}{42}, a_3 \to \frac{8}{65}, a_4 \to \frac{49373}{46838610}, a_5 \to \frac{1}{9}, a_6 \to \frac{49373}{2629536}, a_7 \to \frac{1}{38}, a_8 \to \frac{15}{32}\}\}$

```
TruthTable[MODEL4]
```

p	q	r	var	Pr
T	T	T	a_8	$\frac{15}{32}$
T	T	F	a_5	$\frac{1}{9}$
T	F	T	a_6	$\frac{49\,373}{2\,629\,536}$
T	F	F	a_2	$\frac{5}{42}$
F	T	T	a_7	$\frac{1}{38}$
F	T	F	a_3	$\frac{8}{65}$
F	F	T	a_4	$\frac{49\,373}{46\,838\,610}$
F	F	F	a_1	$\frac{12}{91}$

We can use **EvaluateProbability** to check that this model is a counterexample to the claim in question:

```
EvaluateProbability[{
  Pr[p ⋀ q | r] == Pr[p | r] Pr[q | r],
  Pr[p ⋀ q | ¬ r] == Pr[p | ¬ r] Pr[q | ¬ r],
  Pr[p ⋀ q] == Pr[p] Pr[q]},
 MODEL4]
```

```
{True, True, False}
```

■ Example #4

Hacking's "Odd Question #5" — *Base Rate Fallacy* example:

```
MODEL5 = PrSAT[
  {
   Pr[𝔼 | ℍ] == 0.8,
   Pr[𝔼 | ¬ ℍ] == 0.1,
   Pr[ℍ] == 0.01
  }]
```

```
{{𝔼 → {a2, a4}, ℍ → {a3, a4}, Ω → {a1, a2, a3, a4}}, {a1 → 0.891, a2 → 0.099, a3 → 0.002, a4 → 0.008}}
```

```
TruthTable[MODEL5]
```

𝔼	ℍ	var	Pr
T	T	a_4	0.008
T	F	a_2	0.099
F	T	a_3	0.002
F	F	a_1	0.891

We can use **EvaluateProbability** to calculate the "posterior" Pr(H | E):

```
EvaluateProbability[Pr[ℍ | 𝔼], MODEL5]
```

```
0.0747664
```