On Bayesian Theories of Evidential Support
Normative and Descriptive Considerations[1]

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Abstract

Epistemologists and philosophers of science have often attempted to express formally the impact of a piece of evidence on the credibility of a hypothesis. In this paper we will focus on the Bayesian approach to evidential support. We will propose a new formal treatment of the notion of *degree of confirmation* and we will argue that it overcomes some limitations of the currently available approaches.

1. Rival Bayesian measures of confirmation

Judgments concerning the support that a piece of information brings to a hypothesis are commonly required in scientific research as well as in other domains (medicine, law), and a major aim of a theory of inductive reasoning is to provide a proper foundation to such judgments.

Within the Bayesian approach to inductive reasoning, an attempt to measure degrees of confirmation, or evidential support, should reflect, and extend, a basic *qualitative* view of confirmation (labelled the "classificatory concept of confirmation" by Carnap, 1950/62, pp. 21-22). This view identifies confirmation with an increase in the probability of the hypothesis (conclusion) *h* provided by the piece of information (premise) *e*, neutrality with a lack of impact of *e* on the probability of *h*, and disconfirmation with a decrease of such a probability as an effect of *e*. A common way to convey this distinction is to formalize the confirmation relation by a mathematical

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function $c(e,h)$, depending on probability values concerning e and h, such that the following condition (BC) is satisfied ("BC" stands for "Bayesian confirmation"):[2]

$$(BC) \qquad c(e,h) \begin{cases} > 0 & \text{if } p(h|e) > p(h) & \text{[confirmation]} \\ = 0 & \text{if } p(h|e) = p(h) & \text{[neutrality]} \\ < 0 & \text{if } p(h|e) < p(h) & \text{[disconfirmation]} \end{cases}$$

Condition (BC) does not put any constraint on the values to be assigned to confirmatory arguments (as long as such values are positive) or to disconfirmatory arguments (as long as such values are negative). Indeed, (BC) does not constrain the choice of one single measure of confirmation as the most adequate (a point emphasized by Fitelson, 1999). As a matter of fact, several non-equivalent measures can be, and have been, devised, which map relevant probability values onto a number which is positive iff $p(h|e) > p(h)$, amounts to 0 iff $p(h|e) = p(h)$, and is negative otherwise. *Table 1* provides a catalogue collected from the literature.

Table 1. Alternative Bayesian measures of confirmation.

$D(e,h) = p(h|e) - p(h)$	Carnap (1950/1962)
$S(e,h) = p(h|e) - p(h|\neg e)$	Christensen (1999)
$M(e,h) = p(e|h) - p(e)$	Mortimer (1988)
$N(e,h) = p(e|h) - p(e|\neg h)$	Nozick (1981)
$C(e,h) = p(h\&e) - p(h) \bullet p(e)$	Carnap (1950/1962)
$R(e,h) = \dfrac{p(h|e)}{p(h)} - 1$	Finch (1960)
$G(e,h) = 1 - \dfrac{p(\neg h|e)}{p(\neg h)}$	Rips (2001)
$L(e,h) = \dfrac{p(e|h) - p(e|\neg h)}{p(e|h) + p(e|\neg h)}$	Kemeny & Oppenheim (1952)

[2] A few remarks are in order here. First, confirmation should be properly conceived as a *three*-place relation involving, beyond h and e, the relevant "background knowledge". However, this point will not affect the present discussion. Thus, for the sake of simplicity, and without loss of generality, we will omit background knowledge from our notation. Secondly, throughout the paper we will assume that e and h are *contingent* (i.e., neither logically true nor logically false) and that the probability function p is *regular* (i.e., that $p(x) \neq 0$ unless x is logically false and $p(x) \neq 1$ unless x is logically true). See Festa (1999) and Kuipers (2000) for discussions on how to handle with Bayesian means some limiting cases which have been excluded here.

Quantitative measures of confirmation such as those listed above have the important property of allowing *ordinal* judgments concerning inductive strength, such as: "hypothesis h receives more empirical support by e_1 than by e_2" or "e confirms to a greater extent h_1 than h_2". Two measures of confirmation are *ordinally equivalent* iff they give the same answer to the question which one, if any, has a higher confirmation value in any given pair of arguments.[3] In this respect, *Table 1* provides a representative sample, for most of the Bayesian confirmation measures which we have been able to identify in the literature, but are not included in our list, are ordinally equivalent to some measure which does appear in the list.[4]

2. A first normative requirement

One way to handle the plurality of Bayesian measures of confirmation is to resort to the long standing and traditional view of inductive logic as an "extension" of classical deductive logic (Carnap, 1950/1962). Consider a function v construed on the basis of classical deductive logic and such that, for any argument (e,h), v assigns it the same positive value (e.g., +1) iff $e \models h$, an equivalent value with opposite sign (–1) iff $e \models \neg h$, and value 0 otherwise. The relationships between the logical implication or refutation of h by e and the conditional probability of h given e yields that any Bayesian confirmation measure c agrees with v in the minimal sense that: if $v(e,h)$ is

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[3] Formally, measure c is ordinally equivalent to measure c^* iff, for any pair of arguments (e_1,h_1) and (e_2,h_2), $c(e_1,h_1) >/=/< c(e_2,h_2)$ iff $c^*(e_1,h_1) >/=/< c^*(e_2,h_2)$.

[4] In particular, measures R, G and L in our list are ordinally equivalent to the following well-known measures of confirmation:

$R^*(e,h) = p(h \mid e)/p(h)$	Keynes (1921)
$G^*(e,h) = p(\neg h)/p(\neg h \mid e)$	Gaifman (1979)
$L^*(e,h) = p(e \mid h)/p(e \mid \neg h)$	Good (1950)

It is easy to show that: $R(e,h) = R^*(e,h) - 1$, $G(e,h) = 1 - [1/G^*(e,h)]$, and $L = [L^* - 1]/[L^* + 1]$. Notice that (unlike R, G and L) R^*, G^* and L^* are always positive and identify 1 as the "neutrality" value, thus departing from condition (*BC*) above. A common strategy to have such measures satisfying condition (*BC*) is to apply logarithms (with base > 1) to them. By this strategy, one again obtains measures ordinally equivalent, respectively, with our R, G and L. However, by the use of logarithms, such measures are not defined when $p(h \mid e) = 1$ and/or when $p(h \mid e) = 0$.

Under rather weak assumptions, it can also be proven that *Table 1* includes no pair of ordinally equivalent measures. In fact, there exist quite simple probability models such that the measaures included in *Table 1* yield eight mutually incompatible rankings of inductive strength. For instance, let it be the case that: $p(e_1) = .03$, $p(e_2) = .01$, $p(e_3) = .65$, $p(e_4) = .52$, $p(h_1) = .02$, $p(h_2) = .15$, $p(h_3) = .50$, $p(h_4) = .70$, $p(h_1 \mid e_1) = .32$, $p(h_2 \mid e_2) = .48$, $p(h_3 \mid e_3) = .75$, and $p(h_4 \mid e_4) = .98$. Then the following rankings obtain: $D(e_2,h_2) > D(e_1,h_1) > D(e_4,h_4) > D(e_3,h_3)$; $S(e_3,h_3) > S(e_4,h_4) > S(e_2,h_2) > S(e_1,h_1)$; $M(e_1,h_1) > M(e_3,h_3) > M(e_4,h_4) > M(e_2,h_2)$; $N(e_4,h_4) > N(e_3,h_3) > N(e_1,h_1) > N(e_2,h_2)$; $C(e_3,h_3) > C(e_4,h_4) > C(e_1,h_1) > C(e_2,h_2)$; $R(e_1,h_1) > R(e_2,h_2) > R(e_3,h_3) > R(e_4,h_4)$; $G(e_4,h_4) > G(e_3,h_3) > G(e_2,h_2) > G(e_1,h_1)$; $L(e_1,h_1) > L(e_4,h_4) > L(e_2,h_2) > L(e_3,h_3)$ (computational details omitted).

positive, the same is true of $c(e,h)$; and if $v(e,h)$ is negative, the same is true of $c(e,h)$. Then, consistency with the following principle may be posited as a plausible adequacy requirement for c (here, "*Ex*" stands for the "extrapolation" from the deductive to the inductive domain):

(Ex_1) if $v(e_1,h_1) > v(e_2,h_2)$, then $c(e_1,h_1) > c(e_2,h_2)$

Fulfilment of principle (Ex_1) guarantees that, by c, any conclusively confirmatory argument (e,h) (i.e., such that $e \models h$) is assigned a higher value than any argument which is *not* conclusively confirmatory and any conclusively disconfirmatory argument (e,h) (i.e., such that $e \models \neg h$) is assigned a lower value than any argument which is *not* conclusively disconfirmatory.

Remarkably, it can be proven (see below) that only *one* among the measures of confirmation listed in *Table 1* actually fulfils (Ex_1), i.e., L.[5] It turns out, however, that there is a rather simple way to obtain a measure of confirmation which *does* fulfil (Ex_1) from either D, S, M, N, C, R or G. To see this, let's consider the formulas to which such measures can be reduced when $e \models h$ and when $e \models \neg h$, respectively, as reported in *Table 2*.

Table 2.

Measure	if $e \models h$	if $e \models \neg h$
$D(e,h)$	$p(\neg h)$	$-p(h)$
$S(e,h)$	$p(\neg h)/p(\neg e)$	$-p(h)/p(\neg e)$
$M(e,h)$	$p(e) \cdot [p(\neg h)/p(h)]$	$-p(e)$
$N(e,h)$	$p(e)/p(h)$	$-p(e)/p(\neg h)$
$C(e,h)$	$p(e) \cdot p(\neg h)$	$-p(e) \cdot p(h)$
$R(e,h)$	$p(\neg h)/p(h)$	-1
$G(e,h)$	1	$-p(h)/p(\neg h)$

Now, for each measure in *Table 2*, let's employ the expressions appearing in the second and third column (taken in *absolute value*) to "normalize" degrees of positive and negative evidential support, respectively, as follows:

[5] As a matter of fact, Kemeny & Oppenheim (1952) originally derived measure L from a set of adequacy requirements one of which (p. 309) actually implies (E_1). Fitelson (2006) – also an advocate of confirmation measures ordinally equivalent to L – has labelled *logicality* the following property, which also implies (Ex_1): $c(e,h)$ is maximal (minimal) when $e \models h$ ($e \models \neg h$).

$$D_{norm}(e,h) = \begin{cases} D(e,h)/p(\neg h) & \text{if } p(h|e) \geq p(h) \\ D(e,h)/p(h) & \text{otherwise} \end{cases}$$

$$S_{norm}(e,h) = \begin{cases} S(e,h)/[p(\neg h)/p(\neg e)] & \text{if } p(h|e) \geq p(h) \\ S(e,h)/[p(h)/p(\neg e)] & \text{otherwise} \end{cases}$$

$$M_{norm}(e,h) = \begin{cases} M(e,h)/\{p(e) \bullet [p(\neg h)/p(h)]\} & \text{if } p(h|e) \geq p(h) \\ M(e,h)/p(e) & \text{otherwise} \end{cases}$$

$$N_{norm}(e,h) = \begin{cases} N(e,h)/[p(e)/p(h)] & \text{if } p(h|e) \geq p(h) \\ N(e,h)/[p(e)/p(\neg h)] & \text{otherwise} \end{cases}$$

$$C_{norm}(e,h) = \begin{cases} C(e,h)/[p(e) \bullet p(\neg h)] & \text{if } p(h|e) \geq p(h) \\ C(e,h)/[p(e) \bullet p(h)] & \text{otherwise} \end{cases}$$

$$R_{norm}(e,h) = \begin{cases} R(e,h)/[p(\neg h)/p(h)] & \text{if } p(h|e) \geq p(h) \\ R(e,h) & \text{otherwise} \end{cases}$$

$$G_{norm}(e,h) = \begin{cases} G(e,h) & \text{if } p(h|e) \geq p(h) \\ G(e,h)/[p(h)/p(\neg h)] & \text{otherwise} \end{cases}$$

For our present purposes, two facts deserve emphasis. First of all, standard probability calculus yields that:

$$D_{norm} = S_{norm} = M_{norm} = N_{norm} = C_{norm} = R_{norm} = G_{norm}$$

In fact, at this point, we have one single (new) Bayesian measure of confirmation. From now on, we will label it "*Z*".

The second important fact is that *Z* does fulfil principle (Ex_1) above. The main formal results reported in this paragraph are summarized in the following theorem, and demonstrated in its proof (see Appendix 1):

Theorem 1. *D*, *S*, *M*, *N*, *C*, *R* and *G* (and all confirmation measures ordinally equivalent to any of these) are inconsistent with principle (Ex_1), whereas *L* and *Z* (and all confirmation measures ordinally equivalent to any of these) satisfy principle (Ex_1)

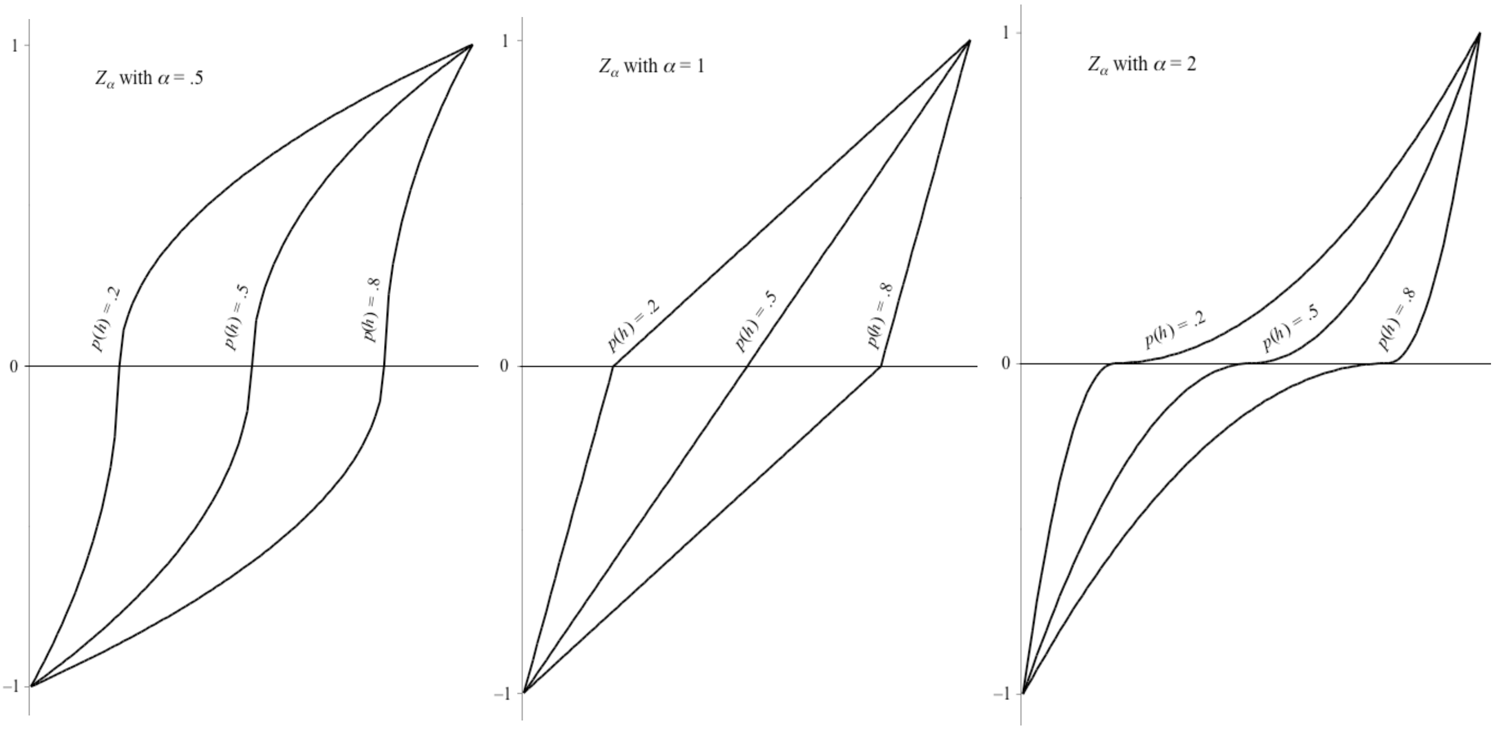
3. *Z*-measures

By means of *Z* it is possible to define a whole set (indeed, a *continuum*) of confirmation measures (we will call them "*Z*-measures") which we would like to introduce as a new account of evidential support within the Bayesian framework. *Z*-measures are defined as follows:

$$Z_\alpha(e,h) = \begin{cases} [Z(e,h)]^\alpha & \text{if } p(h|e) \geq p(h) \\ -|Z(e,h)|^\alpha & \text{otherwise} \end{cases}$$

where α is positive. By being all monotone transformations of the same quantity, all *Z*-measures are ordinally equivalent. α plays the role of a parameter controlling curvature (see *Fig. 1*).

Figure 1. Graphical representations of *Z*-measures with different values of α. On the *x* axis lie posterior probability values of *h*.



As shown by the figure: when $\alpha = 1$, the graphical representation of Z_α is composed by a pair of straight lines; when $\alpha < 1$, Z_α is a "*S*"-shaped function, indicating a higher sensitivity of confirmation (disconfirmation) to relatively small positive (negative) departures from the prior probability of *h*; finally, when $\alpha > 1$, Z_α is an "inverse *S*"-shaped function, indicating a higher sensitivity of confirmation (disconfirmation) to relatively wide positive (negative) departures from the prior probability of *h*.

Theorem 1 above immediately implies that all *Z*-measures fulfil principle (Ex_1). Here are some further desirable properties of *Z*-measures (which are *not*, however, sufficient to establish the superiority of *Z*-measures over all their major competitors):

P_1) If $p(h|e_1) > p(h|e_2)$, then $Z_\alpha(e_1,h) > Z_\alpha(e_2,h)$ (a property involved in the Bayesian solution of the "ravens paradox" provided by Horwich, 1982).

P_2) If $h_1 \models e$, $h_2 \models e$, and $p(h_1) > p(h_2)$, then $Z_\alpha(e,h_1) > Z_\alpha(e,h_2)$ (a property involved in the Bayesian solution of the "grue paradox" provided by Sober, 1994).

P_3) If e confirms h and x is an "irrelevant conjunct" to h with respect to e (i.e.: $p(e|h) = p(e|h\&x)$), then $Z_\alpha(e,h) > Z_\alpha(e,h\&x)$ (a property involved in the Bayesian solution of the paradox of "irrelevant conjunction" provided by Hawthorne & Fitelson, 2006).[6]

4. Symmetries

In what follows we will compare the adequacy of *Z*-measures with that of the measures collected in *Table 1* on both normative and descriptive grounds.

As far as the normative side is concerned, Eells & Fitelson (2002) have recently proposed to narrow down the set of adequate Bayesian candidates for quantifying confirmation by considering a series of "symmetries and asymmetries". We will now present an extended and systematic treatment of this issue.

Consider a set of sentences Γ, closed under negation and conjunction, on which a probability function p is defined.[7] In what follows, by a *symmetry* we will mean a function σ from $\Gamma\otimes\Gamma$ into $\Gamma\otimes\Gamma$ such that $\sigma(e,h)$ is obtained from (e,h) by applying the negation operator (¬) to either e or h (or both) and/or by inverting them. On the whole, there are *seven* such symmetry functions:

$E(e,h) = (\neg e,h)$

$H(e,h) = (e,\neg h)$

[6] Interestingly, Fitelson (2006) has argued in favor of L (and measures ordinally equivalent to L) by noticing its "historical uniqueness" (up to ordinal equivalence) in fulfilling (Ex_1) (in fact, his stronger "logicality" requirement, see footnote 5) as well as property P_1, while dismissing attempts to prove the *mathematical* uniqueness of L (up to ordinal equivalence) because committing "continuity assumptions" concerning the probability function p would be involved. (By the way, Fitelson is also aware that L and ordinally equivalent measures enjoy property P_2 – see his 2001, p. 18 – and Hawthorne & Fitelson, 2006, have proven that they enjoy P_3 as well.) Now, it is an immediate corollary of the main formal result presented in this paper (see *Theorem 2* below) that *Z*-measures are *not* ordinally equivalent to L (nor, for that matter, to any other measure in *Table 1*). Thus, it seems that the definition of *Z*-measures (which *are* consistent with continuity assumptions concerning p and which also demonstrably enjoy Fitelson's "logicality"), provides an existence proof that the uniqueness of L (up to ordinal equivalence) is *just* historical.

[7] Again, strictly speaking, in order to exclude some degenerating cases in the analysis to follow, a few restrictions on Γ and p must be specified: beyond the regularity of p and the contingency of e and h (see footnote 2), it is assumed that there exist $e,h \in \Gamma$ such that $p(e) \neq p(h)$ and $p(e\&h) \neq p(e) \cdot p(h)$ (i.e., that e and h are not probabilistically independent).

$I(e,h) = (h,e)$
$EH(e,h) = (\neg e,\neg h)$
$EI(e,h) = (h,\neg e)$
$HI(e,h) = (\neg h,e)$
$EHI(e,h) = (\neg h,\neg e)$

Here, "*E*" stands for "(negation of the) evidence", "*H*" for "(negation of the) hypothesis", and "*I*" for "inversion (of evidence and hypothesis)".

Now let s (for "sign") be a function from $\Gamma\otimes\Gamma$ into $\{-1, 1\}$, defined as follows: $s(e,h) = 1$ if $p(h|e) \geq p(h)$; $s(e,h) = -1$ if $p(h|e) < p(h)$.

Function s allows a distinction between *convergent* and *divergent* symmetries: a symmetry σ is convergent iff (e,h) and $\sigma(e,h)$ have the same sign, i.e., iff, for any $e,h \in \Gamma$, $s(e,h) = s[\sigma(e,h)]$; a symmetry σ is divergent iff (e,h) and $\sigma(e,h)$ have opposite signs, i.e., iff, for any $e,h \in \Gamma$, $s(e,h) = -s[\sigma(e,h)]$. It can easily be proven that *H*, *E*, *HI* and *EI* are divergent symmetries, whereas *I*, *EH* and *EHI* are convergent.

By means of s, we will also state the following definition:

Def. 1. *c mirrors* a symmetry σ iff, for any $e,h \in \Gamma$,
$s(e,h) \bullet c(e,h) = s(\sigma(e,h)) \bullet c(\sigma(e,h))$

The conditions on which c mirrors any particular symmetry σ immediately follow from *Def. 1*:

Divergent symmetries:
$c(e,h) = -c(E(e,h)) = -c(\neg e,h)$
$c(e,h) = -c(H(e,h)) = -c(e,\neg h)$
$c(e,h) = -c(EI(e,h)) = -c(h,\neg e)$
$c(e,h) = -c(HI(e,h)) = -c(\neg h,e)$

Convergent symmetries:
$c(e,h) = c(I(e,h)) = c(h,e)$
$c(e,h) = c(EH(e,h)) = c(\neg e,\neg h)$
$c(e,h) = c(EHI(e,h)) = c(\neg h,\neg e)$

As we shall see later on, it is also important to distinguish a measure c mirroring a given symmetry in two separate sub-cases, i.e., when (e,h) is a confirmatory (or neutral) *vs*. disconfirmatory argument. The relevant definition is as follows:

Def. 2. c mirrors a symmetry σ *in case of confirmation [disconfirmation]* iff,
for any $e,h \in \Gamma$ such that $s(e,h) = 1$ $[-1]$, $s(e,h) \bullet c(e,h) = s(\sigma(e,h)) \bullet c(\sigma(e,h))$

5. A new set of adequacy requirements

The point of Eells & Fitelson (2002) – whose discussion is in fact limited to *E*, *H*, *I* and *EH*[8] – is that the normative appeal of *c* mirroring any particular symmetry may be assessed by some rather simple and intuitive examples, which different measures of confirmation may or may not match.

To illustrate, take *E*: should it be the case that $c(e,h) = -c(\neg e,h)$?

Consider a standard deck. It seems that having drawn a Jack card confirms that card being a face card to a greater extent that having drawn a card which is *not* a Jack disconfirms the same hypothesis, the reason being that in the former case the available evidence conclusively establishes the hypothesis, whereas in the latter case the available evidence does not conclusively refute the hypothesis. Conversely, having drawn an ace card disconfirms that card being a face card to a greater extent than having drawn a card which is *not* an ace confirms the same hypothesis. On the basis of similar examples, Eells & Fitelson (2002) argue that an acceptable Bayesian measure of confirmation should *not* mirror symmetry *E*.

There is any general and simple principle to determine how a measure of confirmation should behave as far as *all* symmetries are concerned? We think there is one, which, just as (Ex_1) above (see 2.), builds on the traditional view of inductive logic as an "extension" of classical deductive logic. Once again, such principle may be stated by means of function *v*, defined on the basis of classical deductive logic (see 2.):

(Ex_2) *c* mirrors σ in case of confirmation [disconfirmation]
iff *v* mirrors σ in case of confirmation [disconfirmation]

We shall see shortly why the distinction between the two cases (confirmation *vs*. disconfirmation) is important. By now, notice that (Ex_2) is a rather strong condition and, as we shall also see, a quite powerful theoretical tool. We don't know of any explicit and general treatment of it. However, it backs up several arguments circulating in inductive logic and in Bayesian confirmation theory in particular. For the moment, we will posit (Ex_2) as a legitimate guideline for the discussion of the symmetries which an adequate Bayesian measure of confirmation should or should not mirror. In what follows, we will try to establish consistency with (Ex_2) as a compelling *desideratum* through a detailed analysis of its consequences.

Resort to principle (Ex_2) may seem more adequate to identify the symmetries *not* to be mirrored (see, for instance, the Jack and the ace examples above) rather than for the selection of those to be mirrored by *c*.

[8] These are precisely the symmetries originally discussed by Carnap (1950/62, § 67).

However, the principle can be usefully applied in the latter sense as well, for instance as far as *H* is concerned. Indeed, from (Ex_2) it follows that:

$$c(e,h) = -c(e,\neg h)$$

In absence of compelling counterexamples (which we were not able to devise), the equality above can be presented as a plausible extension of the plain logical fact that *e* implies *h* (i.e., $e \models h$) iff *e* refutes $\neg h$ (i.e., $e \models \neg\neg h$).[9]

Now consider symmetry *I*, i.e., the following equality:

$$c(e,h) = c(e,h)$$

Should we require that a confirmation measure classify inversely symmetric arguments as equally strong (recall that the inverse symmetry function is convergent), i.e., that it mirror *I*? Eells & Fitelson (2002) argue that we should not, but interestingly in this connection they explicitly resort *only* to counterexamples involving pairs of *confirmatory* arguments. This is noteworthy, for consider again the ace example above: unlike in the Jack case, we see as intuitively compelling that having drawn an ace card *does* disconfirm that being a face card to the same extent that having drawn a face card disconfirms that being an ace. Thus, we concur with Eells & Fitelson in finding unattractive the above equality *in case of confirmation* but not necessarily *in case of disconfirmation* as well. Indeed, *c* mirroring *I* in case of disconfirmation (but *not* in case of confirmation) also follows from (Ex_2) and, in absence of compelling counterexamples (which we were not able to devise), this seems a plausible extension of the theorem of deductive logic according to which $e \models \neg h$ iff $h \models \neg e$; i.e., a plausible inductive counterpart of the commutative (or "inversely symmetric") nature of logical inconsistency.

Generalizing this line of argument, it can be shown that principle (Ex_2) yields a definite answer for any symmetry both in case of confirmation and in case of disconfirmation. *Table 3* reports the outcomes of such analysis, illustrating in any single case either the relevant parallelism with an analogous state of affairs whithin deductive logic or relevant counterexamples.

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[9] Interestingly, Kemeny & Oppenheim (1952) derived their proposal of measure *L* from a set of requirements which includes mirroring symmetry *H*. They posit such requirement as "natural" and seem to presuppose (Ex_2) in their argument. Eells & Fitelson (2002) take a similar position.

Table 3. The whole set of consequences of (Ex_2).

divergent symmetries

	in case of confirmation		in case of disconfirmation	
E	**NO** for some confirmation (e,h) $c(e,h) \neq -c(\neg e,h)$	e.g.: c(Jack, face) > $-c$(not-Jack, face)	**NO** for some disconfirmation (e,h) $c(e,h) \neq -c(\neg e,h)$	e.g.: c(ace, face) < $-c$(not-ace, face)
H	**YES** for any confirmation (e,h) $c(e,h) = -c(e,\neg h)$	e implies h iff e refutes $\neg h$	**YES** for any disconfirmation (e,h) $c(e,h) = -c(e,\neg h)$	e refutes h iff e implies $\neg h$
EI	**NO** for some confirmation (e,h) $c(e,h) \neq -c(h, \neg e)$	e.g.: c(Jack, face) > $-c$(face,not-Jack)	**YES** for any disconfirmation (e,h) $c(e,h) = -c(h, \neg e)$	e refutes h iff h implies $\neg e$
HI	**YES** for any confirmation (e,h) $c(e,h) = -c(\neg h, e)$	e implies h iff $\neg h$ refutes e	**NO** for some disconfirmation (e,h) $c(e,h) \neq -c(\neg h, e)$	e.g.: c(ace, face) < $-c$(not-face, ace)

convergent symmetries

	in case of confirmation		in case of disconfirmation	
I	**NO** for some confirmation (e,h) $c(e,h) \neq c(h,e)$	e.g.: c(Jack, face) > c(face, Jack)	**YES** for any disconfirmation (e,h) $c(e,h) = c(h,e)$	e refutes h iff h refutes e
EH	**NO** for some confirmation (e,h) $c(e,h) \neq c(\neg e,\neg h)$	e.g.: c(Jack, face) > c(not-Jack, not-face)	**NO** for some disconfirmation (e,h) $c(e,h) \neq c(\neg e,\neg h)$	e.g.: c(ace, face) < c(not-ace, not-face)
EHI	**YES** for any confirmation (e,h) $c(e,h) = c(\neg h, \neg e)$	e implies h iff $\neg h$ implies $\neg e$	**NO** for some disconfirmation (e,h) $c(e,h) \neq c(\neg h, \neg e)$	e.g.: c(ace, face) < c(not-face, not-ace)

6. Why *Z*-measures are normatively appealing

We claim that *Z*-measures should be of interest for any Bayesian confirmation theorist who finds principle (Ex_2) compelling for the plain fact that such measures are all consistent with (Ex_2) whereas, to our knowledge, no alternative measure proposed so far is. The proof of the following theorem (see Appendix 2), which is the main formal result of the present work, illustrates the point.[10]

[10] Here it is appropriate to mention the only previous occurrence of *Z*-measures which we have been able to detect in the literature. Rescher (1958) noticed measure *Z* (i.e., Z_α with $\alpha = 1$) and even displayed its graphical representation. However, on the basis of a set of adequacy requirements which did not include consistency with (Ex_2), Rescher eventually identified $Z(e,h) \cdot p(e)$ as the most adequate *explicatum* for the degree of evidential support. The difference may seem minor, but is not. As a matter of fact, it can be shown that Rescher's measure does *not* share many of the properties of *Z*-measures which have been discussed here and, in particular, is not consistent with (Ex_2).

Theorem 2. *Z*-measures satisfy all the consequences of (Ex_2), whereas *D*, *S*, *M*, *N*, *C*, *R*, *G* and *L* (and all confirmation measures ordinally equivalent to any of these) are inconsistent with (Ex_2).[11]

The fact that measure *L* does fulfil principle (Ex_1) but not (Ex_2) shows that the former does not imply the latter. Notably, it is also not the case that (Ex_2) implies (Ex_1).[12] Thus both principles are complementary but independent constraints stemming from the same idea of confirmation theory as an extension of deductive logic. Arguably, however, one may still cast doubts on this strict parallelism and question why, after all, we should want a formal account of *inductive* reasoning to be consistent with (Ex_2).

Let's take symmetry *I* as an example about which such doubts may typically be raised. Suppose that you must rely on two generally accurate but fallible devices A_1 and A_2 in order to obtain information about a playcard. A_1 works as follows: it classifies a submitted card as a Jack or a not-Jack and reports the outcome with an accuracy rate of 95% (*not* depending on whether the card is or is not, in fact, a Jack). A_2 works as follows: it classifies a submitted card as a face or not-face and reports the outcome, again with an accuracy rate of 95% (*not* depending on whether the card is or is not, in fact, a face). Now suppose a card is drawn from a standard deck by a genuinely random process and submitted to both A_1 and A_2. Let *e* be "A_1 reports that it is a Jack card" and *h* be "A_2 reports that it is a face card". On reflection, it seems clear to us that *e* confirms *h* more than *h* confirms *e*, even if none is strictly implied by the other. (If this example sounds unbearably artificial, let A_1 and A_2 be two medical tests for two diagnostic hypotheses such that the former implies the latter.)

But now consider a third device A_3 such that it classifies a card as an ace or a not-ace, again with an accuracy rate of 95% (*not* depending on whether the card is or is not, in fact, an ace). Suppose a card is selected by a new random extraction and submitted to both A_3 and A_2. Let e^* be "A_3 reports that it is an ace" and *h*, once again, "A_2 reports that it is a face card". Here, we see no reason why the two arguments (e^*,*h*) and (*h*,e^*) should have a different (negative) strength. Sentences e^* and *h* are just "almost" incompatible; and, so it seems, that is all. (If this example sounds

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[11] Interestingly, ordinal equivalence to *Z*-measures is *not* a *sufficient* condition for a confirmation measure to be consistent with (Ex_2). To see why, consider a measure defined as follows: $[Z(e,h)]^\alpha$ if $p(h|e) \geq p(h)$; $-|Z(e,h)|^\beta$ otherwise (where both α and β are positive). Whenever $\alpha \neq \beta$, such measure is ordinally equivalent to all *Z*-measures, but it also violates (Ex_2) (for instance, it does not mirror symmetry *H*).

[12] In fact, Bayesian measures of confirmation can be defined which satisfy (Ex_2) but not (Ex_1). For instance, the following: $Z(e,h)/|C(e,h)|$.

unbearably artificial, let now A_3 and A_2 be two medical tests for two mutually exclusive diagnostic hypotheses.)

The problem is that, unlike Z-measures, all the competitors violating (Ex_2) also violate intuition here by either ranking (e,h) as an equally strong confirmatory argument as (h,e)[13] or ranking one between (e^*,h) and (h,e^*) as a significantly stronger disconfirmatory argument than the other (without even agreeing on *which one* is the strongest!)[14].

Similar examples may be conceived for any row in *Table 2* to underpin the plausibility of (Ex_2) and therefore of Z-measures as a normatively appropriate basis to assess inductive strength.

7. An empirical test of competing measures

One intriguing question in the *psychology* of inductive reasoning is whether the most normatively justified measures of confirmation also are the most descriptively accurate. More generally, the study of reasoning and cognition has often widely profited from the comparison between normative theories (such as probability theory, or expected utility theory) and actual human judgment and behavior.

In a previous work (Tentori *et al.*, 2006) we presented an experiment aiming at an empirical test of some of the alternative Bayesian measures of confirmation. For each of 26 participants, an urn was ramdomly and covertly selected out of *A* (30 black balls + 10 white balls) and *B* (15 black balls + 25 white balls). Participants were then asked to judge the quantitative impact of each outcome of a series of ten random extractions without replacement on the two complementary hypotheses that *A vs*. *B* had been selected (see Tentori *et al.*, 2006, for a detailed description of the procedure). The experiment showed that, when predictions about confirmation judgments were computed from the randomness of the initial selection, the composition of the urns and the outcomes of previous draws, measure *L* outperformed several competitors (including *D*, *N* and *C*), yielding a reliably higher average correlation with the judged impacts on both hypothesis *A* and *B* provided by the participants. In what follows we will test all the competing confirmation measures listed in *Table 1*, along with the simplest Z-measure (i.e., with $\alpha = 1$), against the same data.

[13] In fact, it will be the case that $c(e,h) = c(h,e)$ if c stands for either C or R (see Appendix 2).

[14] In fact, it will be the case that $c(e^*,h) < c(h,e^*)$ if c stands for D, S, G or L, and $c(e^*,h) > c(h,e^*)$ if c stands for M or N (computational details omitted).

As *Table 4* shows, *Z* is the most descriptively accurate measure in this experimental setting. (The table also reports a reliable superiority of the most accurate measures in comparison with posterior probability, taken as a predictor of confirmation judgments. Notice that this indicates that the participants appropriately distinguished posteriors from evidential impact.)

Table 4. Average correlations for competing confirmation measures.

measure (predictor)	average correlation with participants' confirmation judgments for the hypothesis: urn *A* selected	urn *B* selected
Z	.756*	.775*
L	.740*	.754*
N	.730*	.745*
M	.628†	.588
G	.549	.631
R	.619	.557
S	.594	.613
C	.586	.605
D	.573	.589
$\boldsymbol{p(A[B]|e)}$	.488	.508

Reported values are the average of 26 correlations (one per participant) between the confirmation judgments predicted by each measure (on the basis of the randomness of the initial selection, the composition of the urns and the outcomes of previous draws) and the confirmation judgments expressed by each participant. Each correlation involved 10 observations.

$p(A[B]|e)$ denotes $p(A|e)$ or $p(B|e)$ as appropriate.

Comparisons by paired t-test with the average correlation for $p(A[B]|e)$: $^*p < .001$; $^\dagger p = .06$.

Consistent with these results, *Table 5* shows that, by paired t-test, *Z* is superior to all competitors in predicting judged impact on both hypotheses *A* and *B*, with only one major exception (the lack of a statistically reliable difference from *L* for hypothesis *A*). Moreover, a simple non-parametric analysis consisting in counting the subjects (out of 26) for whom *Z* predicted better than each rival measure shows that *Z* is generally a better predictor in the majority of cases (see *Table 5*).

Table 5. Comparison of *Z* with other confirmation measures.

hypothesis: urn A selected

	L	**N**	**M**	**G**	**R**	**S**	**C**	**D**
Z	$t = 1{,}44$ *n.s.*	$t = 1{,}92$ $p = .07$	$t = 2{,}87$ $p = .008$	$t = 4{,}52$ $p = .0001$	$t = 3{,}25$ $p = .003$	$t = 3{,}13$ $p = .004$	$t = 3{,}25$ $p = .003$	$t = 3{,}61$ $p = .001$
	18	18	18	22	21	21	21	21

hypothesis: urn B selected

	L	**N**	**M**	**G**	**R**	**S**	**C**	**D**
Z	$t = 2{,}02$ $p = .05$	$t = 2{,}31$ $p = .03$	$t = 3{,}50$ $p = .002$	$t = 3{,}51$ $p = .002$	$t = 4{,}01$ $p = .0005$	$t = 2{,}84$ $p = .009$	$t = 2{,}96$ $p = .007$	$t = 3{,}28$ $p = .003$
	18	18	22	24	23	19	19	21

Each higher cell reports a paired t-test between the correlations obtained from the measure in the associated column and from measure *Z*. Each lower cell shows the number of participants (out of 26) for whom *Z* predicted better than the rival measure at the top of the column.

These results suggest that the virtues of *Z*-measures might not be confined to the normative level of epistemological reflection, but extend to the descriptive dimension of the psychology of confirmation.

We are not claiming that the probabilistic computations on which Bayesian measures are based should be taken literally as models of the cognitive processes leading to confirmation judgments. We are well aware that, when judging probabilities, subjects often depart from the Bayesian prescriptions (Kahneman, Slovic & Tversky, 1982) and conform to them only under specific conditions (Girotto & Gonzalez, 2001). Precisely for this reason, it is noticeable that a quantitative Bayesian account of evidential impact can reach a considerable predictive accuracy relative to intuitive judgments of confirmation. Whatever the routes by which confirmation judgments are elaborated by naïve subjects, the experiment illustrates that some salient features of such judgments may be captured, to a significant degree, by a normatively justified account of confirmation. In Marr's (1982) terms, such an account might work as an effective, although approximated, computational-level model, compatible with more "realistic" process-level descriptions of the cognitive bases of ordinary inductive reasoning.

In any event, more research is needed to assess the consequences of the present results. In a touchstone study in the psychology of inductive reasoning with statements involving familiar biological categories and "blank" biological predicates (such as "robins use serotonin as a neurotransmitter"), Osherson *et al*. (1990) documented that inversely symmetric confirmatory arguments are *not* generally judged as equally strong, consistent with principle (Ex_2) (see *Table 3*). Further studies aiming at the empirical test of various symmetries and asymmetries might provide evidence about substantial qualitative phenomena concerning inductive reasoning and play a crucial role in assessing the descriptive reliability of competing measures of confirmation outside the context of our urn-based experiment.

In our opinion, the discussion presented here strongly suggests that the interaction between the normative and the descriptive dimension might be fruitful in this field of inquiry as it has been in many other areas of the study of human reasoning.

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APPENDIX 1

Theorem 1. D, S, M, N, C, R and G (and all confirmation measures ordinally equivalent to any of these) are inconsistent with principle (Ex_1), whereas L and Z (and all confirmation measures ordinally equivalent to any of these) satisfy principle (Ex_1).

Proof

Simple probability models prove the first part of the theorem.

Suppose that $p(x) + p(y) + p(z) = 1$ and $p(z) > .5$. (For instance, a fair die is tossed: $x = 1$, $y = 2$, $z =$ greater than 2.) Then let it be the case that: $e_1 = y$; $e_2 = (x \vee y)$; $h_1 = (y \vee z)$; $h_2 = x$. Notice that $e_1 \models h_1$ while $e_2 \not\models h_2$, therefore $v(e_1,h_1) > v(e_2,h_2)$. Moreover, $e_1 \models \neg(\neg h_1)$ while $e_2 \not\models \neg(\neg h_2)$, therefore $v(e_1,\neg h_1) < v(e_2,\neg h_2)$.

Now: $D(e_1,h_1) = p(x)$, while $D(e_2,h_2) = \{p(x)/[p(x) + p(y)]\} - p(x)$. It follows that $D(e_1,h_1) < D(e_2,h_2)$ iff $[p(x) + p(y)] < ½$, and the latter is (by hypothesis) the case.

Since $S(e,h) = D(e,h)/p(\neg e)$, $S(e_1,h_1) < S(e_2,h_2)$ immediately follows from $D(e_1,h_1) < D(e_2,h_2)$ and $p(\neg e_1) > p(\neg e_2)$.

Since $p(e_1) < p(e_2)$ and $p(h_1) > p(h_2)$, it follows that $p(e_1)/p(h_1) < p(e_2)/p(h_2)$. And since $M(e,h) = D(e,h) \bullet [p(e)/p(h)]$ and $D(e_1,h_1) < D(e_2,h_2)$, it follows that $M(e_1,h_1) < M(e_2,h_2)$.

Since $C(e,h) = D(e,h) \bullet p(e)$, $p(e_1) < p(e_2)$ and $D(e_1,h_1) < D(e_2,h_2)$, it follows that $C(e_1,h_1) < C(e_2,h_2)$.

Since h_1 and $\neg h_2$ are equivalent, it follows that $p(h_1) \bullet p(\neg h_1) = p(h_2) \bullet p(\neg h_2)$. Then, since $N(e,h) = C(e,h)/[p(h) \bullet p(\neg h)]$ and $C(e_1,h_1) < C(e_2,h_2)$, it follows that $N(e_1,h_1) < N(e_2,h_2)$.

Since $p(h_1) > p(h_2)$ and $R(e,h) = D(e,h)/p(h)$, $R(e_1,h_1) < R(e_2,h_2)$ follows from $D(e_1,h_1) < D(e_2,h_2)$.

Finally, since $G(e,\neg h) = -R(e,h)$, from $R(e_1,h_1) < R(e_2,h_2)$ follows that $G(e_1,\neg h_1) > G(e_2,\neg h_2)$.

As far as the second part of the theorem is concerned, notice that, if $v(e_1,h_1) > v(e_2,h_2)$, then at least one of the following is true:[15]

a) $p(h_1|e_1) = 1$ and $p(h_2|e_2) < 1$ (equivalently: $p(e_1|\neg h_1) = 0$ and $p(e_2|\neg h_2) > 0$)

b) $p(h_1|e_1) > 0$ and $p(h_2|e_2) = 0$ (equivalently: $p(e_1|h_1) > 0$ and $p(e_2|h_2) = 0$)

Suppose that a) is true. Then: $L(e_1,h_1) = p(e_1|h_1)/p(e_1|h_1) = 1$ and $Z(e_1,h_1) = p(\neg h_1)/p(\neg h_1) = 1$, whereas simple algebraic considerations show that $L(e_2,h_2) < 1$ and $Z(e_2,h_2) < 1$.

Now suppose that b) is true. Then: $L(e_2,h_2) = -p(e_2|\neg h_2)/p(e_2|\neg h_2) = -1$ and $Z(e_2,h_2) = -p(h_2)/p(h_2) = -1$, whereas simple algebraic considerations show that $L(e_1,h_1) > -1$ and $Z(e_1,h_1) > -1$.

Thus, if either a) or b) is true (which is always the case if $v(e_1,h_1) > v(e_2,h_2)$), then both $L(e_1,h_1) > L(e_2,h_2)$ and $Z(e_1,h_1) > Z(e_2,h_2)$

[15] Given that p is regular (see footnote 2).

APPENDIX 2

Theorem 2. *Z*-measures satisfy all the consequences of (Ex_2), whereas *D*, *S*, *M*, *N*, *C*, *R*, *G* and *L* (and all confirmation measures ordinally equivalent to any of these) are inconsistent with (Ex_2).

Proof

In order to prove the theorem we will use three lemmas.

Lemma 1. Any confirmation measure *c* satisfying (*BC*) also satisfies all the consequences of (Ex_2) iff *c* mirrors *H* and *c* mirrors *I* in case of confirmation but not in case of disconfirmation.

Proof. Trivially, if *c* satisfies all the consequences of (Ex_2), then *c* mirrors *H* and *c* mirrors *I* in case of confirmation but not in case of disconfirmation.

On the other hand, suppose *c* does mirror *H* and does mirror *I* in case of confirmation but not in case of disconfirmation. Then, for any $e,h \in \Gamma$ such that $s(e,h) = 1$:

$c(e,h) = -c(e,\neg h)$	[since *c* mirrors *H*]
$-c(e,\neg h) = -c(\neg h,e)$	[since *c* mirrors *I* in case of disconfirmation]
$c(e,h) = -c(\neg h,e)$	therefore: *c* mirrors *HI* in case of confirmation and *EI* in case of disconfirmation
$-c(\neg h,e) = c(\neg h,\neg e)$	[since *c* mirrors *H*]
$c(e,h) = c(\neg h,\neg e)$	therefore: *c* mirrors *EHI* in case of confirmation

Moreover, for some $e,h \in \Gamma$ such that $s(e,h) = 1$:

$c(e,h) \neq c(h,e)$	[since *c* does not mirror *I* in case of confirmation]
$c(h,e) = -c(h,\neg e)$	[since *c* mirrors *H*]
$-c(h,\neg e) = -c(\neg e,h)$	[since *c* mirrors *I* in case of disconfirmation]
$-c(\neg e,h) = c(\neg e,\neg h)$	[since *c* mirrors *H*]
$c(e,h) = -c(e,\neg h)$	[since *c* mirrors *H*]
$c(e,h) \neq -c(h,\neg e)$	therefore: *c* does not mirror *EI* in case of confirmation and does not mirror *HI* in case of disconfirmation
$c(e,h) \neq -c(\neg e,h)$	therefore: *c* does not mirror *E*
$c(e,h) \neq c(\neg e,\neg h)$	therefore: *c* does not mirror *EH*
$c(e,\neg h) \neq c(h,\neg e)$	therefore: *c* does not mirror *EHI* in case of disconfirmation

Lemma 2. *Z* mirrors *H* and *Z* mirrors *I* in case of confirmation but not in case of disconfirmation.

Proof. *Z* mirrors *H*, because, for any $e,h \in \Gamma$ such that $s(e,h) = 1$, $Z(e,h) = [p(h|e) - p(h)]/p(\neg h) = [p(\neg h) - p(\neg h|e)]/p(\neg h) = -[p(\neg h|e) - p(\neg h)]/p(\neg h) = -Z(e,\neg h)$.

Moreover, *Z* mirrors *I* in case of disconfirmation because, for any $e,h \in \Gamma$ such that $s(e,h) = -1$, $Z(e,h) = [p(h|e) - p(h)]/p(h) = [p(h|e)/p(h)] - 1 = [p(e|h)/p(e)] - 1 = [p(e|h) - p(e)]/p(e) = Z(h,e)$.

Finally *Z* does not mirror *I* in case of confirmation because, for any $e,h \in \Gamma$ such that $s(e,h) = 1$, by positing $Z(e,h) = Z(h,e)$ it follows that $p(h)/p(\neg h) = p(e)/p(\neg e)$ (computations omitted), which implies that, whenever $p(h) \neq p(e)$, $Z(e,h) \neq Z(h,e)$.

Lemma 3. If a confirmation measure c satisfying (*BC*) is a function f of Z (i.e. $c(e,h) = f[Z(e,h)]$) such that f is injective and $f(x) = -f(-x)$, then c also satisfies all the consequences of (Ex_2)

Proof. If c is a function of Z, then by definition, $Z(e,h) = Z(h,e)$ implies $c(e,h) = c(h,e)$. Therefore, since Z mirrors I in case of disconfirmation (*Lemma 2*), the same is true of c. Moreover, if c is an *injective* function of Z, then by definition, $Z(e,h) \neq Z(h,e)$ implies $c(e,h) \neq c(h,e)$. Therefore, since Z does not mirror I in case of confirmation (*Lemma 2*), the same is true of c. Finally, if $c(e,h) = f[Z(e,h)]$ and $f(x) = -f(-x)$, then $Z(e,h) = -Z(e,\neg h)$ implies $c(e,h) = -c(e,\neg h)$. Therefore, since Z mirrors H (*Lemma 2*), the same is true of c.
By *Lemma 1*, it follows that if c is a function f of Z such that f is injective and $f(x) = -f(-x)$, then c satisfies all the consequences of (Ex_2).

To prove the theorem, we will now prove that:

(a) any Z-measure is an injective function f of Z such that $f(x) = -f(-x)$ and therefore, by *Lemma 3*, satisfies all the consequences of (Ex_2);

(b) D, S, M, N, C, R, G and L (and all measures of confirmation ordinally equivalent to any of these) are inconsistent with (Ex_2).

Proof of (a). The very definition of Z-measures shows that they are all functions of Z.
Notice that, by principle (*BC*), for any $e_1,e_2,h_1,h_2 \in \Gamma$ such that $s(e_1,h_1) = 1$ and $s(e_2,h_2) = -1$, $Z(e_1,h_1) \neq Z(e_2,h_2)$ and $Z_\alpha(e_1,h_1) \neq Z_\alpha(e_2,h_2)$. Moreover, it immediately follows from the definition of Z-measures that, for any $e_1,e_2,h_1,h_2 \in \Gamma$ such that $s(e_1,h_1) = s(e_2,h_2) = 1$, $Z(e_1,h_1) \neq Z(e_2,h_2)$ implies $Z_\alpha(e_1,h_1) \neq Z_\alpha(e_2,h_2)$ and that, for any $e_1,e_2,h_1,h_2 \in \Gamma$ such that $s(e_1,h_1) = s(e_2,h_2) = -1$, $Z(e_1,h_1) \neq Z(e_2,h_2)$ implies $Z_\alpha(e_1,h_1) \neq Z_\alpha(e_2,h_2)$. This shows that any Z-measure is an injective function of Z.
Finally, let it be the case that $Z(e_1,h_1) = x$ and $Z(e_2,h_2) = -x$. It is now sufficient to notice that $[x]^\alpha = |-x|^\alpha = -[-|-x|^\alpha]$. This shows that any Z-measure is a function f of Z such that $f(x) = -f(-x)$, and completes the proof of (a).

Proof of (b). Consider first D, S, M, N, G and L. Algebraic tranformations yield that, for any $e,h \in \Gamma$ such that $s(e,h) = -1$:

$D(e,h) = Z(e,h) \bullet p(h)$ | $D(h,e) = Z(h,e) \bullet p(e)$
$S(e,h) = Z(e,h) \bullet [p(h)/p(\neg e)]$ | $S(h,e) = Z(h,e) \bullet [p(e)/p(\neg h)]$
$M(e,h) = Z(e,h) \bullet p(e)$ | $M(h,e) = Z(h,e) \bullet p(h)$
$N(e,h) = Z(e,h) \bullet [p(e)/p(\neg h)]$ | $N(h,e) = Z(h,e) \bullet [p(h)/p(\neg e)]$
$G(e,h) = Z(e,h) \bullet [p(h))/p(\neg h)]$ | $G(h,e) = Z(e,h) \bullet [p(e))/p(\neg e)]$
$L(e,h) = Z(e,h)/\{1 - [p(h) - p(\neg h)] \bullet [1 + Z(e,h)]\}$ | $L(e,h) = Z(h,e)/\{1 - [p(e) - p(\neg e)] \bullet [1 + Z(h,e)]\}$

By *Lemma 2*, for any $e,h \in \Gamma$ such that $s(e,h) = -1$, $Z(e,h) = Z(h,e)$. Then, whenever $p(h) \neq p(e)$, each of the above measures (as well as any measure of confirmation ordinally equivalent to any of them) assigns a higher value to either (e,h) or (h,e), thus failing to mirror I in case of disconfirmation, contrary to (Ex_2).

Now consider the remaining measures C and R. For any $e,h \in \Gamma$ such that $s(e,h) = 1$, $C(e,h) = p(h\&e) - p(h) \bullet p(e) = C(h,e)$ and $R(e,h) = [p(h|e)/p(h)] - 1 = [p(e|h)/p(e)] - 1 = R(h,e)$. Then, both C and R (as well as any measure of confirmation ordinally equivalent to any of them) mirror I in case of confirmation, contrary to (Ex_2).

This completes the proof of (b), and of the theorem.