The Logical Generality of Williamson's "Fitch-Like" Modal Triviality

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1 Setting the stage: initial version of the triviality

In a recent paper, Timothy Williamson [4] makes the following passing remark.[2]

> The principle that every truth is possibly necessary can now be shown to entail that every truth is necessary by a chain of elementary inferences in a perspicuous notation unavailable to Hegel.

In the early part of this note, I will understand the modal triviality Williamson has in mind as the trivialization of certain propositional modal logics by the addition of $\ulcorner p \to \sim\Box\sim\Box p \urcorner$ as an *axiom schema*. By *trivialization*, I mean that the logic (expanded in this way) has as a *theorem schema* $\ulcorner p \to \Box p \urcorner$.[3] In the final section of this note, I will look at the case in which we add another modal operator ($\Diamond$) to the language, and we consider adding an axiom schema of the form $\ulcorner p \to \Diamond\Box p \urcorner$. This is a distinct problem (in general), since (presumably) some modal logicians out there will be inclined to reject the interchangeability of $\ulcorner \Diamond p \urcorner$ and $\ulcorner \sim\Box\sim p \urcorner$. We'll see that *without* this assumption, much of the triviality's generality is lost (independently of what other modal axioms are present and independently of what underlying propositional logic we presuppose).

In this note, I will (1) lay out a general framework for asking questions about the logical generality of Williamson's triviality, and (2) use that framework to pose and answer a variety of such questions. The use of automated reasoning techniques will be essential to my approach, which will be purely axiomatic, making no appeal to Kripke semantics (or any other modal semantics).

[1] **Draft**. Comments welcome. Thanks to Brian Weatherson and the participants of his (12/09/04) blog thread "Defending Williamson" (especially, Kenny Easwaran) for a stimulating discussion that inspired this note. This note has lead to some interesting open questions in non-standard (and non-normal) modal logics, and to some interesting applications of automated reasoning tools. I have used Otter (theorem prover) and Paradox (model finder) to find all proofs and models. The details of these computational investigations will be reported later in the Appendix to this paper.

[2] I'm calling Williamson's triviality "Fitch-like" because it *seems* similar in form to Fitch's paradox. But, Fitch's paradox differs from Williamson's in that it involves *two kinds* of modal operators (one epistemic and one alethic). That is, Fitch's paradox arises from the assumption that "all truths are possibly (in an *alethic* sense) necessary (in an *epistemic* sense)". Williamson, I take it, intends no ambiguity in the modality of his statement that "all truths are possibly necessary". So, I will interpret $\Diamond$ and $\Box$ as expressing the same *kind* of modality (I guess the intended interpretation is *alethic*). This note concerns only the formal, logical generality of Williamson's triviality, and not such interpretive questions. But, the logical simplicity of (*i.e.*, the lack of modal *ambiguity* in) Williamson's triviality does make it easier to study from our logical point of view. The reader is cautioned not to infer anything much about *Fitch*'s paradox [1] from what we say here.

[3] Along the way, I will briefly discuss other senses of trivialization, having to do with the addition of an axiom '$p \to \sim\Box\sim\Box p$' rather than the axiom schema, or the addition of a rule schema $\ulcorner \vdash p \Rightarrow \vdash \sim\Box\sim\Box p \urcorner$ (or a rule) rather than an axiom schema (or an axiom).

2 Our Logical Framework

Our framework is purely axiomatic. It involves Hilbert-style axiomatic versions of the salient class of logics. Since Williamson's triviality (as we understand it initially) involves only implication (→), negation (~), and the necessity operator (□), it will suffice for our purposes to begin with underlying implication-negation propositional logics, axiomatized in the usual Hilbert-style, as follows:

Rule of Inference. Modus Ponens (MP): If $\vdash p$ and $\vdash \ulcorner p \to q \urcorner$, then $\vdash q$.[4]
Axiom Schemas. Some set of axiom schemas (in $\langle \to, \sim \rangle$) which, with (MP), is sufficient to derive all the theorem schemas of the $\langle \to, \sim \rangle$-logic in question.[5]

Next, I will introduce several propositional $\langle \to, \sim \rangle$-logics that will serve as the underlying logics of our subsequent modal logics.

2.1 Some underlying propositional $\langle \to, \sim \rangle$-Logics

C. Classical propositional $\langle \to, \sim \rangle$-logic consists of (MP) together with the following three axiom schemas (or any (MP)-equivalent set):

C_1: $\vdash \ulcorner (p \to q) \to ((q \to r) \to (p \to r)) \urcorner$

C_2: $\vdash \ulcorner p \to (\sim p \to q) \urcorner$

C_3: $\vdash \ulcorner (\sim p \to p) \to p) \urcorner$

H. Intuitionistic propositional $\langle \to, \sim \rangle$-logic consists of (MP) together with the following four axiom schemas (or any (MP)-equivalent set):

H_1: $\vdash \ulcorner p \to (q \to p) \urcorner$

H_2: $\vdash \ulcorner (p \to (q \to r)) \to ((p \to q) \to (p \to r)) \urcorner$

H_3: $\vdash \ulcorner (p \to \sim p) \to \sim p \urcorner$

H_4: $\vdash \ulcorner p \to (\sim p \to q) \urcorner$

Ł. Łukasiewicz infinite-valued propositional $\langle \to, \sim \rangle$-logic consists of (MP) together with the following four axiom schemas (or any (MP)-equivalent set):

Ł$_1$: $\vdash \ulcorner p \to (q \to p) \urcorner$

Ł$_2$: $\vdash \ulcorner (p \to q) \to ((q \to r) \to (p \to r)) \urcorner$

[4] It is customary in Hilbert-style calculi to treat rules of inference as *theoremhood* preserving, not "truth" preserving. I will do this uniformly, for *all* rules of inference (not just the necessitation rule, which *must* be treated this way – see below). Since I'll be using automated reasoning techniques, the rule of (MP) is actually *condensed detachment* (CD), in which all substitution instances are *most general* [2]. Since all the logics I discuss here are *D*-complete [3], this results in no loss of generality.

[5] I won't discuss here what happens if we add further connectives to our logics.

Ł$_3$: $\vdash\ulcorner((p \to q) \to q) \to ((q \to p) \to p)\urcorner$

Ł$_4$: $\vdash\ulcorner(\sim p \to \sim q) \to (q \to p)\urcorner$

R. The relevant propositional $\langle\to,\sim\rangle$-logic **R** consists of (MP) together with the following six axiom schemas (or any (MP)-equivalent set):

R_1: $\vdash\ulcorner p \to p\urcorner$

R_2: $\vdash\ulcorner(p \to q) \to ((r \to p) \to (r \to q))\urcorner$

R_3: $\vdash\ulcorner(p \to (q \to r)) \to (q \to (p \to r))\urcorner$

R_4: $\vdash\ulcorner(p \to (p \to q)) \to (p \to q)\urcorner$

R_5: $\vdash\ulcorner(p \to q) \to (\sim q \to \sim p)\urcorner$

R_6: $\vdash\ulcorner\sim\sim p \to p\urcorner$

M. The relevant propositional $\langle\to,\sim\rangle$-logic **M** consists of (MP) together with the following four axiom schemas (or any (MP)-equivalent set):

M_1: $\vdash\ulcorner(p \to q) \to ((r \to p) \to (r \to q))\urcorner$

M_2: $\vdash\ulcorner(p \to \sim p) \to \sim p\urcorner$

M_3: $\vdash\ulcorner(p \to \sim q) \to (q \to \sim p)\urcorner$

M_4: $\vdash\ulcorner\sim\sim p \to p\urcorner$

Note: **Ł** and **R** are extensions of **M**, which is a "minimal" relevant $\langle\to,\sim\rangle$-logic.[6]

2.2 Some Modal Logics

We can obtain various *modal* propositional $\langle\to,\sim\rangle$-logics by adding to any of our propositional $\langle\to,\sim\rangle$-logics the following additional rule and axiom schema(s) concerning the relation between $\Box$ and the logical connectives $\langle\to,\sim\rangle$:

Modal Rule of Inference. Necessitation (N): If $\vdash p$, then $\vdash\ulcorner\Box p\urcorner$.
Modal Axiom Schemas. Some set of modal axiom schemas involving $\to$, $\sim$, and $\Box$.[7] The modal axiom schemas that I will consider are the following.

K: $\vdash\ulcorner\Box(p \to q) \to (\Box p \to \Box q)\urcorner$

D: $\vdash\ulcorner\Box p \to \sim\Box\sim p\urcorner$

[6]Relevant logicians typically use the names "**TW**" or "**P**$-$**W**" for our minimal relevant logic **M**.

[7]Remember, for now, I am assuming the interchangeability of $\ulcorner\Diamond p\urcorner$ and $\ulcorner\sim\Box\sim p\urcorner$. This is why I state all axiom schemas below in terms of $\to$, $\sim$, and $\Box$. I will weaken this assumption later, and see what happens when $\Diamond$ is added (*without* assuming that $\ulcorner\Diamond p\urcorner$ and $\ulcorner\sim\Box\sim p\urcorner$ are interchangeable). For obvious reasons, I am not discussing logics containing the axiom schema $\vdash\ulcorner\Diamond\Box p \to \Box p\urcorner$.

B: $\vdash \ulcorner p \to \Box{\sim}\Box{\sim}p \urcorner$

T: $\vdash \ulcorner \Box p \to p \urcorner$

4: $\vdash \ulcorner \Box p \to \Box\Box p \urcorner$

5: $\vdash \ulcorner {\sim}\Box{\sim}p \to \Box{\sim}\Box{\sim}p \urcorner$

By combining an underlying $\langle \to, {\sim} \rangle$-logic with some combination of the modal axiom schemas, one can obtain various modal propositional $\langle \to, {\sim} \rangle$-logics. For instance, **CKT45** yields the classical modal logic **S5** (as does **CKT5**). We can also combine non-classical $\langle \to, {\sim} \rangle$-logics with modal schemas. But, when we do this, we need to be careful about thinking of them as analogous to the "corresponding" classical modal logics, since we are (for now) assuming that $\ulcorner \Diamond p \urcorner$ and $\ulcorner {\sim}\Box{\sim}p \urcorner$ are interchangeable, which may be rejected by some proponents of such non-classical logics. So as not to beg any questions, I will just refer to these logics by their acronyms (*e.g.*, **HKT5** is what you get when you add axioms **K**, **T**, and **5** to the underlying intuitionistic $\langle \to, {\sim} \rangle$-logic **H**). In the next section, I examine the consequences of adding Williamson's formula:

W: $\vdash \ulcorner p \to {\sim}\Box{\sim}\Box p \urcorner$

to the modal logics that can be cooked-up from the above ingredients.

3 The triviality phenomenon and its scope

The following outline summarizes what I know about the triviality phenomenon, assuming the interchangeability of $\ulcorner \Diamond p \urcorner$ and $\ulcorner {\sim}\Box{\sim}p \urcorner$. See Appendix for proofs.

1. Logics *without* the **K** axiom:
   (a) Classical logics:
      i. Known to be non-trivial:
         A. **CDBT4**
      ii. Known to be trivial:
         A. **C5**. Remark: any $\langle \to, {\sim} \rangle$-logic **X** with the following three theorem schemas is such that **X5** is trivial (*e.g.*, **M**):
            $\vdash \ulcorner (p \to q) \to ((r \to p) \to (r \to q)) \urcorner$
            $\vdash \ulcorner {\sim}{\sim}p \to p \urcorner$
            $\vdash \ulcorner (p \to {\sim}q) \to (q \to {\sim}p) \urcorner$
            This includes all of our underlying $\langle \to, {\sim} \rangle$-logics, *except* **H**.
      iii. Open:
         - None.
   (b) Non-Classical Logics
      i. Known to be non-trivial:

A. **H**: **HDBT45**. *No intuitionistic non-**K** logics are trivial*.
B. **Ł**: **ŁDBT4**.
C. **R**: **EDBT4**.
D. **M**: **MDBT4**.

ii. Known to be trivial:
A. **H**: None. *No intuitionistic non-**K** logics are trivial*.
B. **Ł**: *Only* logics extending **Ł5**.
C. **R**: *Only* logics extending **R5**.
D. **M**: *Only* logics extending **M5**.

iii. Open:
- None.

2. Logics *with* the **K** axiom:

(a) Classical logics:

i. Known to be non-trivial:
A. **CKDB**.

ii. Known to be trivial:
A. **CKT**, **CK4**, and, of course, **CK5**.

iii. Open:
- None.

(b) Non-Classical Logics

i. Known to be non-trivial:
A. **H**: **HKDBT45**. *No intuitionistic **K** logics are trivial*.
B. **Ł**: **ŁKTD4** and **ŁKB**.
C. **R**: **RKTD4** and **RKB**.
D. **M**: **MKTD4** and **MKB** (of course).

ii. Known to be trivial:
A. **H**: None. *No intuitionistic **K** logics are trivial*.
B. **Ł**: **ŁKTDB4**, and, of course, **ŁK5**.
C. **R**: **RKTDB4**, and, of course, **RK5**.
D. **M**: **MKTDB4**, and, of course, **MK5**.

iii. Open:
A. **H**: None. *No intuitionistic **K** logics are trivial*.
B. **Ł**: **ŁKTB**.
C. **R**: **RKTB**.
D. **M**: **MKTB**.

But, what if we do *not* assume the interchangeability of $\ulcorner \Diamond p \urcorner$ and $\ulcorner {\sim}\Box{\sim}p \urcorner$?

4 The interchangeability of $\ulcorner\Diamond p\urcorner$ and $\ulcorner{\sim}\Box{\sim}p\urcorner$

In our set-up of the triviality so far, we re-wrote all the modal axiom schemas involving $\Diamond$ in terms of $\sim$ and $\Box$, by assuming that $\ulcorner\Diamond p\urcorner$ and $\ulcorner{\sim}\Box{\sim}p\urcorner$ are interchangeable. What happens if we weaken this assumption, and add a new modal operator $\Diamond$ to the logic(s), with no (or weaker) assumptions about their relationship? The only schemas that change are **W**, **D**, **B**, and **5**, which become:

$\mathbf{W}_\Diamond$: $\ulcorner p \to \Diamond\Box p\urcorner$

$\mathbf{D}_\Diamond$: $\ulcorner \Box p \to \Diamond p\urcorner$

$\mathbf{B}_\Diamond$: $\ulcorner p \to \Box\Diamond p\urcorner$

$\mathbf{5}_\Diamond$: $\ulcorner \Diamond p \to \Box\Diamond p\urcorner$

If this is the *only* change we make, and we assume *nothing* about the relationship between $\Diamond$ and $\Box$, then (obviously) the triviality *disappears completely*. Of course, if we add "bridge principles" about the connection between $\Box$ and $\Diamond$, then we can make the triviality more general. The strongest such principle is interchangeability. But, there are weaker principles along these lines. For instance, I take it that even intuitionists would be happy with the axiom scheme $\ulcorner\Diamond p \to {\sim}\Box{\sim}p\urcorner$ (I think it's the converse they may not like). If we add this as an axiom scheme to our $\Diamond/\Box$ modal logics, then we get a more general triviality phenomenon, but not as general as if we assume interchangeability. In particular, logics extending $\mathbf{CK5}_\Diamond$ are trivial in the presence of this weaker bridge principle. But, now the **K** axiom is required for triviality (*i.e.*, not even $\mathbf{CTBD45}_\Diamond$ is trivial any longer). And, non-classically, the only logics that are trivial under this weaker assumption are extensions of $\mathbf{MK5}_\Diamond$.

To sum up: the triviality Williamson has in mind is, for the most part, a classical phenomenon. And, non-classically, it (for the most part) only arises if we assume the interchangeability of $\ulcorner\Diamond p\urcorner$ and $\ulcorner{\sim}\Box{\sim}p\urcorner$, and the **5** axiom. Moreover, the paradox seems *never* to arise in intuitionistic modal logics.

Appendix: Technical Details

An Appendix detailing all proofs, models, and automated reasoning techniques underlying the above report is in preparation. The Appendix will also discuss the triviality-ramifications further assumptions about $\Box$ and $\Diamond$.

References

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