

New Short(est) Single Axioms for Classical Logic*

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Abstract

We report new short(est) single axioms for classical sentential logic. Under condensed detachment, we prove that Meredith’s 21-symbol C/N axiom and six new axioms are shortest among double-negation-free formulas with at most six distinct variables. We also give six 19-symbol axioms in each of C/f and C/o ; in each case, shortestness reduces to one unresolved 17-symbol representative. In the equational setting, we report short single axioms in four primitive signatures, including a shortest 17-symbol C/f identity, a new 17-symbol C/N identity, and a new 20-symbol O/N single axiom (together with its A/N dual), improving the previous O/N bound. We conjecture that every single axiom reported here is shortest, although some shorter candidates remain uneliminated in several signatures, and the lists at the displayed lengths may not be complete. A Zenodo archive with Python code for systematically searching for short single axioms (in both kinds of systems), machine-checkable proof certificates for each of our axioms, and the Prover9 and Vampire 5.0.1 source releases used in the reference verifications accompanies the paper.

Keywords: single axioms; classical sentential logic; condensed detachment; automated reasoning; logical matrices; equational logic.

Introduction

The paper has two parts. Section 1 concerns formulas used as axioms with condensed detachment as the sole rule of inference. Section 2 concerns identities used as axioms in equational logic, with substitution and replacement of equals as the primitive rules. Although both questions ask how much of classical logic can be compressed into one short expression, the proof systems and the corresponding minimality questions are distinct.

1 Condensed-detachment single axioms

We use Polish notation throughout the main text. In the proof in Appendix A, we switch to explicit parenthesized notation because it makes the term structure more transparent.

*This paper is dedicated to the memory of Bill McCune (1953–2011), Ted Ulrich (1940–2020), and Larry Wos (1930–2020).

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In Polish notation, the binary connective C is implication, N is negation, and f and o are the constants falsum and verum. The weight of a formula is its total number of connective, constant, and variable occurrences; we write $\text{wt}(\alpha)$ for the weight of α . Our sole rule is *condensed detachment* (CD): after renaming variables apart, from $C\alpha\beta$ and γ infer $\sigma\beta$, where σ is a most general unifier of α and γ [1, 5].

Here a classical tautology is a *single axiom* exactly when it derives the corresponding complete basis displayed in Theorem 1 by CD, up to renaming variables. This criterion agrees with ordinary axiomhood under substitution and detachment: every consequence under those rules is a substitution instance of a CD consequence [1, Theorem 1][14, p. 100], and the displayed basis formulas have no proper tautological generalizations, as verified below.

Our claims regarding C/N single axioms and candidates concern only *double-negation-free formulas with at most six distinct variables*. We call this the restricted C/N class. A formula is double-negation free (DN-free) when it has no subformula $NN\beta$.

Ulrich [14, p. 113] asked whether Meredith’s 21-symbol C/N axiom and his 19-symbol C/f axioms are shortest, and noted the analogous question for Meredith’s 19-symbol C/o axioms [10, 11]. Meredith himself wrote, “I have found no other 21-letter (C, N) axiom” [10, p. 155]. Our contributions and the remaining open questions are summarized in Table 1. By *same-length classification* we mean determining all single axioms of a specified length, up to renaming variables.

language	length	new axioms	lower bound
C/N	21	A_1 – A_6	proved in restricted class; classification open
C/f	19	A_7 – A_{12}	open; same-length classification also open
C/o	19	A_{13} – A_{18}	open; same-length classification also open

Table 1: Condensed-detachment results and lower-bound status. The C/N claims concern DN-free formulas with at most six distinct variables.

Among these condensed-detachment results, the six new C/N axioms are known to be shortest in the restricted C/N class. Without that restriction, they are shortest known. The twelve 19-symbol C/f and C/o formulas are verified single axioms, but their minimality depends, respectively, on the unresolved 17-symbol candidates F_{17} and O_4 below. In all three signatures, completeness of the list at the displayed length also remains open: further single axioms of that same length may exist.

$$\begin{aligned}
A_1 &= CCpCCNpqrCsCCNtCrtCpt, \\
A_2 &= CpCCqCprCCNrCCNstqCsr, \\
A_3 &= CpCCNqCCNrscptCCtqCrq, \\
A_4 &= CpCCNqCCNrsc tqCCrtCrq, \\
A_5 &= CCpqCCCrCstCqCNsNpCps, \\
A_6 &= CCCpqCCCNrNsrtCCtpCsp;
\end{aligned}$$

$$\begin{aligned}
A_7 &= CCCCpCCqrfstCCtqCpq, \\
A_8 &= CCCCpfqrCCCrstCtpCtr, \\
A_9 &= CCCpqCrstCCsCCftpCrp, \\
A_{10} &= CCpqCCCrCstCqCqfCps, \\
A_{11} &= CCpqCCCrCstCqCpfCps, \\
A_{12} &= CCpqCCCCCqrfsCtpCtq;
\end{aligned}$$

$$\begin{aligned}
A_{13} &= CCpqCCCrCstCqCosCps, \\
A_{14} &= CCpqCCCCoprCCstpCtq, \\
A_{15} &= CCCpqCrstCCsCopCtCrp, \\
A_{16} &= CCCpqrCCCCrspbCosCts, \\
A_{17} &= CCpCCqrCosCCsqCtCpq, \\
A_{18} &= CCpqCCCCoqrCCstpCtq.
\end{aligned}$$

Theorem 1. *Each of A_1 – A_6 derives by condensed detachment the Łukasiewicz C/N basis*

$$CCpqCCqrCpr, \quad CCNppp, \quad CpCNpq.$$

Each of A_7 – A_{12} derives by condensed detachment the Tarski–Bernays C/f basis

$$CCpqCCqrCpr, \quad CpCqp, \quad CCCpqppp, \quad Cfp.$$

Each of A_{13} – A_{18} derives by condensed detachment the Łukasiewicz C/o basis

$$L_4 = CCCpqrCCrpCsp, \quad L_5 = o.$$

Consequently, every A_i displayed above is a condensed-detachment single axiom for classical logic in its respective signature.

Proof. A truth-table calculation verifies that every displayed A_i is a classical tautology. For each A_i , Prover9 [8] verifies a condensed-detachment derivation of the conjunction of the corresponding basis members from A_i alone. Hints guide the search but are not additional premises. The three displayed bases are complete for classical logic under substitution and detachment in their respective signatures, so the claim follows. $\square$

1.1 Search and verification method

Our computational archive [9] contains the current search toolkit, a preserved snapshot of Walsh’s legacy generator, the official Prover9 and Vampire 5.0.1 sources used for the reference verifications, equational search statistics and survivor ledgers, and machine-checkable proof certificates. The legacy representative tables were generated earlier and are audited here, not newly rerun. Their recorded matrix assignments can be checked individually; we do not claim a new bulk verification of all of them. Some retained historical transcripts use Vampire 4.5.1, as their version banners and the candidate index specify. The 5.0.1 sources are not

claimed to reproduce those older transcripts byte for byte. The complete Prover9 basis proofs are independently replayable with the archived official release. Our search results can be located through the archive’s `CANDIDATE_INDEX.md`, which links named candidates to their elimination certificates, records earlier labels, and points to the larger candidate tables and survivor lists. The following four stages summarize the methodology implemented in the accompanying code. By *general filters* below we mean the truth-table, stock-matrix, and Vampire countermodel or completed-saturation screens.

For the condensed-detachment searches, a theorem of Łukasiewicz [2] substantially reduces the lengths that must be checked. If α is a single axiom of length n , then $Cz\alpha$, where z is fresh, is a single axiom of length $n + 2$. To see this using only CD, take variable-disjoint copies $Cz\alpha$ and $Cw\alpha'$. The major antecedent z unifies with the entire second formula. Since z does not occur in α , the conclusion is α , up to renaming. Consequently, the greatest shorter length in each admissible parity suffices: a still shorter single axiom would generate one at that frontier. This padding argument is specific to the condensed-detachment setting.¹ Thus our equational lower-bound searches check every smaller syntactically admissible weight unless a signature-specific argument permits one to be skipped. In the C/f signature, *e.g.*, every term has odd weight, and hence so does every identity $t = x$; only the odd weights are syntactically admissible. But, for the other equational signatures, both even weights and odd weights are admissible.

1. For each target length, generate the formula structures, or *resonators*, and then their variable-identification instances [15].
2. Compute the truth table of the most general formula for each resonator. The false rows determine exactly which variable identifications can remain tautological. This gives the tautological candidates without separately recomputing a full truth table for every instance.
3. Eliminate candidates with a stock of previously found finite models. For each survivor, ask Vampire [3] for a model (or a saturated set of clauses, as in [4]) in which the candidate and condensed detachment hold while a known complete basis fails. Every such model is a certificate that the candidate is insufficient.
4. Apply deeper proof or model search to the small remainder, using various theorem provers (both off the shelf ones, like Vampire, as well as special purpose programs we have written for condensed detachment and paramodulation). A candidate is reported as a single axiom only when a known complete basis has been derived. It is reported as shortest only when every shorter candidate has also been eliminated.

¹There is no corresponding general padding theorem for equational one-bases, which in the current setting must be of the form $t = x$, with a single variable on one side, by the constant-operation argument given below. For a counterexample to a general equational +2 padding rule, consider the variety defined by $u(x) = x$ in a signature with one unary operation. It has a one-basis of weight 4 but none of weight 6. A defining equation must have a variable on one side: otherwise every constant unary operation would satisfy it. A weight-6 candidate is therefore $u(u(u(v))) = x$. If $v = x$, a three-element algebra in which u is a three-cycle satisfies the candidate but not $u(x) = x$; if $v \neq x$, the candidate is not even valid in the target variety, as witnessed by the two-element algebra in which u is the identity and v and x receive different values.

1.2 Lower bounds and elimination results

Two facts explain why representative lists suffice. First, an elimination certificate transfers from a formula to all its substitution instances. A matrix or a model certified by completed saturation satisfies CD and every instance of the representative while a classical target fails. Every instance of a specialization is already an instance of the original formula, so the same certificate eliminates the specialization.

Second, CD derivations *lift* from specialized axioms to their generalizations. Induct on the derivation: at an inference, the original unifier supplies a common instance of the lifted antecedent and minor premise. A most general unifier therefore produces a conclusion generalizing the original conclusion. The lifted major premise cannot be a bare variable, since all formulas concerned are classical tautologies. Thus it has the required implication form. This proves the induction step without adding a substitution rule to CD. Each displayed basis formula has no proper tautological generalization, apart from renaming variables. We checked this finite fact by replacing subterms by variables, allowing repeated new variables only at identical subterms, and evaluating the truth tables of all resulting generalizations. Hence, if a specialized candidate derives one of these bases, lifting gives that exact basis from the representative.

The finite searches and the two appendix arguments establish the following lower-bound and elimination results. In the restricted C/N class (DN-free, at most six distinct variables), the legacy weight-20 candidates have been eliminated, and the weight-19 analysis reduces the remaining question to the representative

$$c_{19,13} = CCpCCqrCNsNrCCrpCrs.$$

Appendix A eliminates this last representative. For C/f , one 17-symbol candidate remains unresolved:

$$F_{17} = CCCCCpqfrsCCspCtp.$$

For C/o , our initial search left the following two unresolved representatives

$$\begin{aligned} O_4 &= CCCpqCorCsCCrpCtp, \\ O_5 &= CCpqCrCCCqsCopCtq. \end{aligned}$$

Appendix B eliminates O_5 . This leaves O_4 as the sole unresolved C/o lower-bound representative.

Theorem 2 (CD lower bound and reduction). *There is no double-negation-free single axiom for classical C/N with at most six distinct variables and weight less than 21 under condensed detachment. Consequently, Meredith’s 21-symbol C/N axiom and A_1 – A_6 are shortest within this restricted class. Their unrestricted shortestness remains open.*

If classical C/f has a single axiom of weight less than 19 under condensed detachment, then F_{17} is a single axiom. Thus Meredith’s 19-symbol C/f axioms and A_7 – A_{12} are shortest if and only if F_{17} is not a single axiom.

Proof. The padding argument above reduces the C/N claim to weights 19 and 20: a shorter single axiom of odd weight would give a 19-symbol single axiom, and one of even weight

would give a 20-symbol single axiom. Padding preserves double-negation freeness. Here we use the legacy DN-free enumerations, which did not impose a fixed variable cap: fresh variables introduced in this reduction are therefore permitted even when the original formula has six variables. A six-variable frontier check alone would not justify this padding reduction. The legacy enumeration retains the most general tautological variable-identification representatives. The N -containing weight-19 cases are obtained from fresh-variable-padded weight-21 candidates in those records. With z fresh, $Cz\alpha$ is such a representative if and only if α is. Formulas without N cannot derive $CCNppp$ by CD, since CD introduces no absent primitive symbol. Formulas that are not classical tautologies cannot be single axioms for classical logic. The countermodel and saturation filters eliminate all the remaining formulas except $c_{19,13}$ at weight 19. Theorem 9 in Appendix A proves that $c_{19,13}$ has no CD consequence of weight below 19. In particular, it derives none of the displayed C/N basis members, whose weights are 11, 6, and 6. Lifting also rules out every specialization of this representative.

This establishes the stated restricted C/N lower bound, attained by Meredith’s axiom [10] and, by Theorem 1, by A_1 – A_6 .

Every C/f formula has odd weight, since its only connective is binary. Padding therefore takes any shorter single axiom to weight 17. Among weight-17 formulas containing f and at least one variable, the legacy enumeration retains the most general tautological instances; every such tautology is a variable-identification instance of a retained entry. Formulas lacking f cannot derive Cfp by CD, and variable-free formulas have only variable-free CD consequences. Countermodels eliminate all retained entries except F_{17} . If an instance of a retained formula were a single axiom, lifting and the maximal-generality fact above would give the exact Tarski–Bernays basis from that retained formula. Hence any weight-17 single axiom would make F_{17} a single axiom. Conversely, if F_{17} is a single axiom, its weight 17 prevents a 19-symbol axiom from being shortest. The sufficiency proofs in Theorem 1 and Meredith’s earlier proofs [10, 11] give the stated biconditional. $\square$

Theorem 3 (*C/o elimination result*). *If classical C/o has a single axiom of weight less than 19 under condensed detachment, then O_4 is a single axiom. Consequently, the 19-symbol single axioms A_{13} – A_{18} are shortest if and only if O_4 is not a single axiom.*

Proof. The completed screen of all weight-17 C/o formulas containing o and at least one variable leaves O_4 , $O_4[t := s]$, O_5 , and $O_5[t := r]$. A formula lacking o cannot derive L_5 by CD, and a variable-free formula cannot derive L_4 by CD. Theorem 10 in Appendix B supplies a seven-element matrix that eliminates O_5 and its instance $O_5[t := r]$. If $O_4[t := s]$ were a single axiom, lifting would give the displayed C/o basis from O_4 . The only remaining sufficiency question for the lower bound is therefore that of O_4 itself. Every C/o formula has odd weight. If a single axiom of weight less than 19 existed, padding would produce a 17-symbol single axiom, which would make O_4 a single axiom by the preceding reduction. Hence ruling out O_4 would rule out every shorter single axiom. Conversely, if O_4 is a single axiom, its weight 17 prevents any 19-symbol axiom from being shortest. Since Theorem 1 establishes the sufficiency of A_{13} – A_{18} , the biconditional follows. $\square$

1.3 Open questions

The unresolved C/f lower-bound candidate is

$$F_{17} = CCCCCpqfrsCCspCtp.$$

Neither its sufficiency nor its insufficiency has been established. Until it is settled, Meredith's 19-symbol C/f axioms and A_7 – A_{12} are shortest known, rather than proved shortest.

The only unsettled 17-symbol C/o lower-bound representative is

$$O_4 = CCCpqCorCsCCrpCtp.$$

Its sufficiency remains open, as does that of its retained variable-identification instance. Until this is settled, the 19-symbol C/o axioms should not be described as shortest.

Within the DN-free, at-most-six-variable C/N class, the lower bound is settled, but completeness of the list at weight 21 remains open. Unrestricted shortestness is not claimed. Several other 21-symbol candidates in this restricted class have not yet been proved or eliminated by independently checkable countermodels. One current example is

$$CCCPNqrCCCCsNstCrqCsq.$$

This cannot undermine the 21-symbol lower bound within the restricted class, but if it is a single axiom (which we doubt) it belongs on the list of shortest C/N axioms within that class. The corresponding same-length classification questions also remain open in the other signatures.

2 Equational single axioms for classical logic

Meredith and Prior describe equational versions of propositional logic in which equations stand at the same level as asserted formulas and the rules are substitution and interchangeability of equals [12]. We follow that presentation and continue to use Polish notation: C is implication, A is conjunction, O is disjunction, N is negation, and f is falsum. An identity is a single axiom for classical logic in a given signature when it is valid in the intended two-element algebra and its equational consequences include a known complete basis in that signature. Before the present work, the shortest known equational single axiom in the O/N signature was due to McCune et al. [6] (weight 22 under the same counting convention). Following McCune et al., we present O/N explicitly and obtain the A/N results by Boolean duality. Every axiomhood, basis, weight, and lower-bound claim below carries over, mutatis mutandis, to A/N : Boolean duality interchanges O and A , leaves N fixed, and preserves equational consequence and weight [7, pp. 3–4]. Thus the A/N results require no independent candidate search. As far as we know, equational single axioms in the other two signatures considered here— C/f and C/N —have not previously been reported.² Our search produced

²None of the languages studied here contains the Sheffer stroke. Stephen Wolfram found the length-15 candidates that include shortest Sheffer-stroke equational single axioms [16, pp. 808–811, 1174]. Our results involve other sets of connectives, and are theoretically independent of those results [6, p. 3].

the following three identities:

$$\begin{aligned} E_{C/\mathfrak{f}} &= CCCCpqrCCCr\mathfrak{f}q\mathfrak{f}\mathfrak{f} = r, \\ E_{O/N} &= ONOpONqrNONqNOOrps = q, \\ E_{C/N} &= CCCCpCqrsqNCsNq = s. \end{aligned}$$

For equations, weight counts every symbol occurrence in the two terms and one occurrence of equality. The respective weights of these identities are 17, 20, and 17. With their usual two-valued interpretations, these equations present the same algebraic semantics that can be formulated in terms of Boolean algebras. We nevertheless call them equational single axioms for classical logic, following Meredith and Prior and emphasizing the logical systems in which the search questions arose.

Theorem 4. *Each of $E_{C/\mathfrak{f}}$, $E_{O/N}$, and $E_{C/N}$ is an equational single axiom for classical logic in its displayed signature.*

Proof. A truth-table calculation verifies each displayed identity in the intended two-element algebra. For $C/\mathfrak{f}$, the target is Meredith’s two-equation basis [12, §3(1), p. 221, $\{C, 0\}$], with his constant 0 written as $\mathfrak{f}$:

$$\begin{aligned} CCp\mathfrak{f}Cqr &= CCrpCqp, \\ CCpqp &= p. \end{aligned}$$

A single Prover9 replay derives their conjunction from $E_{C/\mathfrak{f}}$ with reported proof length 77. For O/N , the target is Meredith’s two-equation basis [12, §3(1), p. 221, $\{A, N\}$], translated into our notation.³

$$\begin{aligned} ONONpqp &= p, \\ ONONpqOrq &= OqOrp. \end{aligned}$$

A single Prover9 replay derives their conjunction from $E_{O/N}$ with reported proof length 191. Their duals,

$$ANANpqp = p, \quad ANANpqArq = AqArp,$$

form the corresponding two-equation A/N basis. Dualizing the candidate and the derivation therefore gives the A/N result without a separate search. Finally, for C/N , the target is Meredith’s two-equation basis [12, §3(1), p. 221, $\{C, N\}$], with its first equation corrected by Prior’s corrigendum [13, p. 452].⁴

$$\begin{aligned} CNpCqr &= CCrpCqp, \\ CCpqp &= p. \end{aligned}$$

³Meredith writes A for disjunction and abbreviates Cpq as $ANpq$. Expand that abbreviation in his first equation $ApAqr = CCrpAqp$, write O for his A , cyclically rename p, q, r as q, r, p , and reverse the equality to obtain our second equation. His second axiom, $CCpqp = p$, gives our first equation.

⁴The 1968 printing has $CpCqp$ on the left of the first equation. Prior’s 1969 corrigendum explicitly replaces it by $CNpCqr$.

A single Prover9 replay derives their conjunction from $E_{C/N}$ with reported proof length 175. These numbers are the proof lengths reported by Prover9, which include bookkeeping and closing inferences; they are not counts of paramodulation inferences alone.

In every Prover9 replay just described, the displayed candidate is the sole non-denial assumption; proof hints guide clause selection but are not premises. Since each displayed candidate derives a complete basis, all three are single axioms in their respective equational calculi. $\square$

2.1 Lower bounds and elimination results

The equational searches check every smaller admissible weight. To state their outcome precisely, let $S_{C/N}$ be the union of the archived C/N survivor lists at weights 15 and 16, and let $S_{O/N}$ be the union of the O/N survivor lists at weights 17, 18, and 19.

Signature	displayed weight	results at shorter weights
C/f	17	none at any admissible weight below 17
C/N	17	none below 15; 15 (7); 16 (16)
O/N	20	none below 17; 17 (12); 18 (44); 19 (295)

Table 2: Survivors of the equational screens below the displayed single axioms; parenthesized numbers give the counts at each weight.

Theorem 5 (Equational lower bounds and eliminations).

1. *There is no equational single axiom for classical C/f of weight less than 17. Consequently, $E_{C/f}$ is shortest.*
2. *Every equational single axiom for classical C/N of weight less than 17 must, up to renaming variables and interchanging the sides of equality, belong to $S_{C/N}$. This set has 23 members: seven of weight 15 and sixteen of weight 16. In particular, there is no such single axiom of weight less than 15.*
3. *Every equational single axiom for classical O/N of weight less than 20 must, up to the same conventions, belong to $S_{O/N}$. This set has 351 members: twelve of weight 17, forty-four of weight 18, and 295 of weight 19. In particular, there is no such single axiom of weight less than 17. The corresponding statements for A/N hold with the Boolean duals of these sets.*

Proof. The constant-operation argument of [6, Lemma 1] applies to each signature here. On a two-element set, interpret every primitive operation, including f when present, by one fixed element d . Every nonvariable term then evaluates to d , so any identity with neither side a variable holds in this algebra. Yet a displayed basis identity with a variable alone on one side fails when that variable receives the other element. Hence such an identity cannot be a single axiom. It therefore suffices to consider identities $t = x$, allowing an interchange of the two sides. At each smaller admissible weight, the enumerator generates every term structure

and every identification of its variable occurrences, including identifications with the right-hand variable. The truth-table filter removes identities that fail in the intended two-element algebra. Each subsequent elimination supplies a countermodel to a known complete basis, or establishes the satisfiability of that countermodel problem by completed saturation. Either outcome rules out the candidate as a single axiom.

For C/f , all candidates at weights 5, 7, 9, 11, 13, 15 are eliminated; smaller identities either omit a required primitive symbol or fail in the two-element algebra. These are all the smaller admissible cases, since C/f identities $t = x$ have odd weight. Together with Theorem 4, this proves (1).

For C/N , the completed screens cover weights 6 through 16. There are no survivors below 15, and their only survivors at weights 15 and 16 are the listed members of $S_{C/N}$. For O/N , the completed screens cover weights 6 through 19. There are no survivors below 17, and their only survivors at weights 17 through 19 are the listed members of $S_{O/N}$. An identity of weight less than 6 cannot contain both required primitive operations. It therefore leaves at least one operation unconstrained and cannot axiomatize the intended nontrivial equational theory. Thus every possible shorter single axiom must occur in the corresponding survivor set, proving (2) and the O/N part of (3). Boolean duality preserves both equational consequence and weight, and maps the O/N basis to the A/N basis. It therefore transfers the same eliminations and survivor sets to A/N . $\square$

The remaining candidates in Theorem 5 prevent shortestness claims for $E_{C/N}$ and $E_{O/N}$. We report these identities only as new *short* single axioms. None of the unresolved candidates has been proved to be a single axiom. We believe that none is sufficient, but this assessment is merely a conjecture.

A Eliminating the last candidate in the restricted 19-symbol C/N search

We prove that the CD closure of

$$T_0 = c_{19,13} = CCpCCqrCNsNrCCrpCr s$$

is a single chain of formulas whose weights are at least 19. Since the classical theorem Cpp has weight 3, this eliminates the candidate. We switch now from Polish notation to explicit parenthesized notation, for readability. Write $T_0 = C(L_0, R_0)$, where

$$\begin{aligned} L_0 &= C(p, C(C(q, r), C(Ns, Nr))), \\ R_0 &= C(C(r, p), C(r, s)). \end{aligned}$$

For a term A , let $\text{con}(A)$ be its number of connective occurrences, and let $\text{occ}_v(A)$ be the number of occurrences of the variable v in A . Call $C(L, R)$ *left-heavy* when

$$\text{con}(L) > \text{con}(R) \quad \text{and} \quad \text{occ}_v(L) \geq \text{occ}_v(R) \quad \text{for every variable } v.$$

Lemma 6. *If $C(L, R)$ is left-heavy, then $\text{wt}(\sigma L) > \text{wt}(\sigma R)$ for every substitution σ .*

Proof. For every term A ,

$$\text{wt}(\sigma A) = \text{con}(A) + \sum_v \text{occ}_v(A) \text{wt}(\sigma v).$$

The strict connective inequality supplies a positive summand, and every variable-occurrence difference is nonnegative. Therefore $\text{wt}(\sigma L) > \text{wt}(\sigma R)$. $\square$

Define $T_{n+1} = \text{CD}(T_0, T_n)$. Direct unification of T_0 with itself gives

$$(1) \quad T_1 = C(C(x, H_1), C(x, z)), \quad H_1 = C(x, C(C(y, Nz), C(NNx, NNz))).$$

For $n \geq 1$, suppose that

$$(2) \quad T_n = C(L_n, R_n) = C(C(x_n, H_n), C(x_n, z_n)).$$

Take a variable-disjoint copy of T_0 , with variables a, b, c, d . Its antecedent and consequent are

$$\begin{aligned} B &= C(a, C(C(b, c), C(Nd, Nc))), \\ D &= C(C(c, a), C(c, d)). \end{aligned}$$

Unifying T_n with B gives

$$a \mapsto L_n, \quad x_n \mapsto C(b, c), \quad z_n \mapsto C(Nd, Nc).$$

Let θ_n denote the last two substitutions. Applying the unifier to D yields

$$(3) \quad T_{n+1} = C(C(c, \theta_n L_n), C(c, d)).$$

Thus T_{n+1} again has the form (2), with $x_{n+1} = c$, $z_{n+1} = d$, and $H_{n+1} = \theta_n L_n$. Moreover, x_{n+1} occurs in H_{n+1} because x_n occurs in L_n and $\theta_n(x_n) = C(b, c)$.

Lemma 7. *Every $T_n = C(L_n, R_n)$ is left-heavy.*

Proof. For T_0 , the connective counts are 6 and 3. The variable multiplicities in L_0 are $p : 1, q : 1, r : 2, s : 1$, whereas those in R_0 are $p : 1, q : 0, r : 2, s : 1$. Hence T_0 is left-heavy. Equation (1) gives $\text{con}(L_1) = 10$ and $\text{con}(R_1) = 1$; the multiplicities in L_1 are $x : 3, y : 1, z : 2$, whereas those in R_1 are $x : 1, z : 1$. Thus T_1 is left-heavy.

Assume that T_n , where $n \geq 1$, is left-heavy. By (3),

$$L_{n+1} = C(c, \theta_n L_n), \quad R_{n+1} = C(c, d).$$

The term $\theta_n L_n$ contains the connective in $\theta_n(x_n) = C(b, c)$, so $\text{con}(L_{n+1}) > \text{con}(R_{n+1}) = 1$. The outer occurrence of c in L_{n+1} matches the occurrence of c in R_{n+1} . Since $\text{occ}_{z_n}(L_n) \geq \text{occ}_{z_n}(R_n) = 1$, the substitution $\theta_n(z_n) = C(Nd, Nc)$ supplies at least one occurrence of d in L_{n+1} , matching its occurrence in R_{n+1} . Every other variable occurs zero times in R_{n+1} or is supplied by the same substitutions. Hence T_{n+1} is left-heavy. The result follows by induction. $\square$

Proposition 8. *Up to renaming variables, the CD closure of T_0 is exactly $\{T_n : n \geq 0\}$.*

Proof. Using T_0 as major and T_n as minor produces T_{n+1} by definition and the unification above. It remains to show that no T_i with $i \geq 1$ can be used as major.

Fix $i \geq 1$ and $j \geq 0$, and take a variable-disjoint copy of T_i . Its antecedent has the form $C(x_i, H_i)$, where x_i occurs properly in H_i . If $T_j = C(L_j, R_j)$ unified with this antecedent, some substitutions σ and σ' would satisfy

$$(4) \quad \sigma(L_j) = \sigma'(x_i), \quad \sigma(R_j) = \sigma'(H_i).$$

Since x_i occurs properly in H_i ,

$$\text{wt}(\sigma'(H_i)) > \text{wt}(\sigma'(x_i)).$$

Equation (4) would therefore imply $\text{wt}(\sigma R_j) > \text{wt}(\sigma L_j)$, contradicting Lemmas 6 and 7. Hence $\text{CD}(T_i, T_j)$ is undefined whenever $i \geq 1$. $\square$

Theorem 9. *The formula $c_{19,13}$ is not a single axiom for classical C/N .*

Proof. The weights of the first chain members are

$$19, 20, 36, 58, 86, 120, 160, \dots$$

For $n \geq 1$, equations (2)–(3) give

$$\text{wt}(T_n) = \text{wt}(L_n) + 4, \quad \text{wt}(T_{n+1}) = \text{wt}(\theta_n L_n) + 6.$$

The substitution θ_n replaces variables by compound terms, so $\text{wt}(\theta_n L_n) \geq \text{wt}(L_n)$. Therefore $\text{wt}(T_{n+1}) \geq \text{wt}(T_n) + 2$. Every formula in the closure has weight at least 19, while the displayed C/N basis members have weights 11, 6, and 6. By Proposition 8, none is derivable from $c_{19,13}$ by condensed detachment. Thus $c_{19,13}$ is not a single axiom. Lifting rules out its substitution instances as well: a derived basis member would lift to a tautological generalization of no greater weight, contradicting the same lower bound of 19. In particular, C_{pp} , of weight 3, is also absent from the closure. $\square$

This is the final individual elimination used in Theorem 2.

B A seven-element matrix eliminating O_5

Let the matrix have domain $V = \{1, 2, 3, 4, 5, 6, 7\}$ and designated set $D = \{3, 6\}$. Interpret $\circ$ as 3 and implication by the following table, with antecedents indexing rows and consequents indexing columns:

$$(5) \quad \begin{array}{c|ccccccc} i & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 7 & 6 & 6 & 6 & 6 & 6 & 7 \\ 3 & 1 & 4 & 6 & 1 & 4 & 3 & 7 \\ 4 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 5 & 6 & 6 & 6 & 6 & 6 & 6 & 7 \\ 6 & 4 & 2 & 6 & 5 & 4 & 6 & 7 \\ 7 & 2 & 2 & 6 & 2 & 2 & 6 & 6 \end{array}$$

Theorem 10. *The formula $O_5 = CCpqCrCCCqsC\text{op}Ctq$ is not a single axiom for classical C/o .*

Proof. In rows 3 and 6 of (5), the designated entries occur only in columns 3 and 6. Therefore

$$(6) \quad x \in D \text{ and } i(x, y) \in D \implies y \in D.$$

Thus condensed detachment preserves designation. Exhaustive evaluation of the $7^5 = 16,807$ assignments to the five variables of O_5 gives only designated values: 5,510 assignments give value 3 and 11,297 give value 6. Every substitution instance of O_5 is therefore designated. By induction on the length of a CD derivation, every CD consequence of O_5 is designated.

Under the assignment $(p, q, r, s) = (2, 1, 1, 3)$, the basis formula $L_4 = CCCpqrCCrpCsp$ evaluates to 7, which is undesigned. Hence L_4 is not a CD consequence of O_5 . Since every single axiom for classical C/o must derive L_4 , the formula O_5 is not a single axiom. Notice that $L_5 = o$ has the designated value 3; separating the two basis targets is essential. $\square$

References

- [1] J. A. Kalman, “Condensed detachment as a rule of inference,” *Studia Logica* 42 (1983), 443–451.
- [2] J. Łukasiewicz, “The shortest axiom of the implicational calculus of propositions,” *Proceedings of the Royal Irish Academy, Section A* 52 (1948), 25–33.
- [3] L. Kovács and A. Voronkov, “First-order theorem proving and Vampire,” in *Computer Aided Verification*, LNCS 8044, Springer, 2013, 1–35. https://doi.org/10.1007/978-3-642-39799-8_1.
- [4] B. Fitelson and N. Peltier, “Applying saturation-based theorem proving to open problems in positive implicational logic,” *Journal of Automated Reasoning* 70 (2026), article 5. <https://doi.org/10.1007/s10817-026-09752-1>.
- [5] W. McCune and L. Wos, “Experiments in automated deduction with condensed detachment,” in *Automated Deduction—CADE-11*, LNCS 607, Springer, 1992, 209–223.
- [6] W. McCune, R. Veroff, B. Fitelson, K. Harris, A. Feist, and L. Wos, “Short single axioms for Boolean algebra,” *Journal of Automated Reasoning* 29 (2002), 1–16. <https://www.cs.unm.edu/~mccune/papers/basax/>.
- [7] G. Birkhoff, *Lattice Theory*, revised ed., American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, 1948.
- [8] W. McCune, *Prover9 and Mace4*, 2009. <https://www.cs.unm.edu/~mccune/prover9/>.
- [9] B. Fitelson, M. J. Walsh, and OpenAI Codex, *New Short(est) Single Axioms for Classical Logic: Search Code and Proof Archive*, version 1.1.0, Zenodo, 2026. <https://doi.org/10.5281/zenodo.23443554>.

- [10] C. A. Meredith, “Single axioms for the systems (C, N) , (C, O) and (A, N) of the two-valued propositional calculus,” *Journal of Computing Systems* 1 (1953), 155–164.
- [11] C. A. Meredith and A. N. Prior, “Notes on the axiomatics of the propositional calculus,” *Notre Dame Journal of Formal Logic* 4 (1963), 171–187. <https://doi.org/10.1305/ndjfl/1093957574>.
- [12] C. A. Meredith and A. N. Prior, “Equational logic,” *Notre Dame Journal of Formal Logic* 9 (1968), 212–226. <https://doi.org/10.1305/ndjfl/1093893457>.
- [13] A. N. Prior, “Corrigendum to C. A. Meredith’s and my paper: ‘Equational logic’,” *Notre Dame Journal of Formal Logic* 10 (1969), 452. <https://doi.org/10.1305/ndjfl/1093893794>.
- [14] D. Ulrich, “A legacy recalled and a tradition continued,” *Journal of Automated Reasoning* 27 (2001), 97–122. <https://doi.org/10.1023/A:1010683508225>.
- [15] L. Wos, “The resonance strategy,” *Computers & Mathematics with Applications* 29 (1995), 133–178.
- [16] S. Wolfram, *A New Kind of Science*, Wolfram Media, 2002. <https://www.wolframscience.com/nks/>.